

MORPHISMS OF CARTAN CONNECTIONS

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ABSTRACT. We define what we call *morphisms* of Cartan connections. We generalize the main theorems on Cartan connections to theorems on morphisms. Many of the known constructions involving Cartan connections turn out to be examples of morphisms. We prove some basic results concerning completeness of Cartan connections. We provide a new method to prove completeness of Cartan connections using families of morphisms.

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Date: December 3, 2009.

1991 *Mathematics Subject Classification.* MA53A40.

Thanks to Charles Frances for pointing out a serious mistake in a prior version of this paper.

1. INTRODUCTION

Definition 1. Let $H \subset G$ be a closed subgroup of a Lie group, with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. A G/H -geometry (also known as a *Cartan geometry* modelled on G/H) on a manifold M is a principal right H -bundle $E \rightarrow M$ and a 1-form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$ called the *Cartan connection*, which satisfies the following conditions:

- (1) Denote the right action of $h \in H$ on $e \in E$ by $r_h e$. The Cartan connection transforms in the adjoint representation:

$$r_h^* \omega = \text{Ad}_h^{-1} \omega.$$

- (2) $\omega_e : T_e E \rightarrow \mathfrak{g}$ is a linear isomorphism at each point $e \in E$.
- (3) For each $A \in \mathfrak{g}$, define a vector field \vec{A} on E (called a *constant vector field*) by the equation $\vec{A} \lrcorner \omega = A$. For $A \in \mathfrak{h}$, the vector fields \vec{A} generate the right H -action:

$$\vec{A}(e) = \left. \frac{d}{dt} r_{e^{tA}} e \right|_{t=0}, \text{ for all } e \in E.$$

All statements in this paper hold equally true for real or holomorphic Cartan geometries with obvious modifications.

Example 1. The bundle $G \rightarrow G/H$ is a G/H -geometry, called the *model*, with Cartan connection $\omega = g^{-1} dg$ the left invariant Maurer–Cartan 1-form on G .

Example 2. Suppose that $\rho : H \rightarrow \text{GL}(n, \mathbb{R})$ is a representation, and let $G = H \rtimes \mathbb{R}^n$. Write elements of G as “matrices”,

$$g = \begin{pmatrix} h & x \\ 0 & 1 \end{pmatrix},$$

for $h \in H$ and $x \in \mathbb{R}^n$. Then the Maurer–Cartan 1-form is

$$g^{-1} dg = \begin{pmatrix} h^{-1} dh & h^{-1} dx \\ 0 & 0 \end{pmatrix}.$$

Definition 2. The *universal constant vector field* of a Cartan geometry $E \rightarrow M$ is the vector field X on $E \times \mathfrak{g}$ given by $X(e, A) = (\vec{A}, 0)$. The *constant flow* of a Cartan geometry is the flow of the universal constant vector field.

Example 3. Returning to example 2, the constant flow is

$$e^{tX} \left(\begin{pmatrix} h & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \right) = \left(\begin{pmatrix} h e^{tA} & x + h \left(\int e^{tA} dt \right) v \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \right).$$

The projected curves in $G/H = \mathbb{R}^n$ are the flow lines of linear vector fields, up to translation. In particular, if we take $G/H = \mathbb{R}^2$, for various choices of H we get:

H	curves in \mathbb{R}^2
$\{1\}$	lines
$\text{SO}(2)$	circles and lines
$\mathbb{R}^+ \text{SO}(2)$	circles, lines and exponential spirals

Definition 3. We will say that a Cartan geometry is *complete* if all of the constant vector fields are complete i.e. if the flow of the universal constant vector field is complete.

Example 4. The model is complete, as the constant vector fields are the right invariant vector fields on G .

Lemma 1 (Sharpe [36] p. 188, theorem 3.15). The Cartan connection of any Cartan geometry $\pi : E \rightarrow M$ determines isomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \pi'(e) & \longrightarrow & T_e E & \longrightarrow & T_m M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{h} \longrightarrow 0 \end{array}$$

for any points $m \in M$ and $e \in E_m$; thus

$$TM = E \times_H (\mathfrak{g}/\mathfrak{h}).$$

Definition 4. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are two G/H -geometries, with Cartan connections ω_0 and ω_1 , and X is manifold, perhaps with boundary and corners. A smooth map $\phi_1 : X \rightarrow M_1$ is a *development* of a smooth map $\phi_0 : X \rightarrow M_0$ if there exists a smooth isomorphism $\Phi : \phi_0^* E_0 \rightarrow \phi_1^* E_1$ of principal H -bundles identifying the 1-forms ω_0 with ω_1 .

Development is an equivalence relation. For example, a development of an open set is precisely a local isomorphism. The graph of Φ is an integral manifold of the Pfaffian system $\omega_0 = \omega_1$ on $\phi_0^* E_0 \times E_1$, and so Φ is the solution of a system of (determined or overdetermined) differential equations, and conversely solutions to those equations determine developments. By lemma 1 the developing map ϕ_1 has differential ϕ_1' of the same rank as ϕ_0' at each point of X .

Definition 5. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are G/H -geometries. Suppose that $\phi_1 : X \rightarrow M_1$ is a development of a smooth map $\phi_0 : X \rightarrow M_0$ with isomorphism $\Phi : \phi_0^* E_0 \rightarrow \phi_1^* E_1$. By analogy with Cartan's method of the moving frame, we will call e_0 and e_1 *frames* of the development if $\Phi(e_0) = e_1$.

Definition 6. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are G/H -geometries. We will say that M_1 *rolls freely* on curves in M_0 to mean that every curve $\phi_0 : C \rightarrow M_0$ has an unramified covering which admits a development with any chosen frames $e_0 \in \phi_0^* E_0$ and $e_1 \in E_1$.

Note that we get to pick $e_1 \in E_1$ arbitrarily, and then we can construct a development $\phi_1 : C \rightarrow M_1$ with e_1 lying in $\phi_1^* E_1$.

Definition 7. We can similarly define the notion of rolling freely on some family of curves, for example immersed curves.

Remark 1. We obtain the same development if we replace the frames by $e_0 h$ and $e_1 h$, for any $h \in H$. The curve mentioned could be a real curve, or (if the manifolds and Cartan geometries are complex analytic) a complex curve, with boundary and corners and arbitrary complex analytic singularities. The frames determine the development, and any frames yield a local development, because the equation $\omega_0 = \omega_1$ is a system of first order ordinary differential equations in local coordinates.

Remark 2. Ehresmann [14] called a Cartan geometry *normal* if it rolls freely on all curves in its model. Kobayashi [22] called such a Cartan geometry *complete*. Sharpe [36] called a Cartan geometry *complete* if it has complete flow. Kobayashi [22] claimed that normalcy and complete flow are equivalent. (This would explain Sharpe’s use of the term *complete*, since he was aware of Kobayashi’s paper.) Clifton [9] proved that normalcy implies complete flow, but he also gave explicit examples to show that complete flow need not imply normalcy—these concepts are *inequivalent*, i.e. Kobayashi’s claim is wrong. Unfortunately this is not well known in the Cartan geometry literature. I will use the word *complete* to mean complete flow, following Sharpe, since Sharpe’s book is now the standard reference work on Cartan connections. There is no way to avoid the unfortunate confusion that fogs the literature.

Example 5. A Riemannian geometry is a Cartan geometry modelled on Euclidean space. As a Riemannian manifold, the unit sphere rolls freely on curves in the plane. Rolling in this context has its obvious intuitive meaning (see Sharpe [36] pp. 375–390). The development of a geodesic on the plane is a geodesic on the sphere, so a portion of a great circle. The upper half of the unit sphere does *not* roll freely on the plane (by our definition), because if we draw any straight line segment in the plane of length more than π , we can’t develop all of it to the upper half of the sphere. As we will see, every Riemannian manifold rolls on curves in a given Riemannian manifold M just when M is complete as a metric space.

Theorem 1. Let M be a manifold with a Cartan geometry. The following are equivalent:

- (1) M rolls freely on curves in its model.
- (2) M rolls freely on curves in any Cartan geometry with the same model.
- (3) M rolls freely on curves in some Cartan geometry with the same model.

Remark 3. Clifton [9] gives examples of Cartan geometries rolling freely on all smooth immersed curves in the model, but not on all smooth curves.

Theorem 2. Let M_1 be a manifold with a Cartan geometry. The following are equivalent:

- (1) M_1 rolls freely on immersed curves in its model.
- (2) M_1 rolls freely on immersed curves in any Cartan geometry with the same model.
- (3) M_1 rolls freely on immersed curves in some Cartan geometry with the same model.

This article begins by studying coframings; starting in section 9 on page 20, we move on to study Cartan geometries. We define a notion of morphism of Cartan geometries, and show that the main theorems on Cartan geometries have analogues for morphisms. For example, the Frobenius–Gromov theorem is generalized to the category of Cartan geometry morphisms. We also develop a theory of morphism deformations, and of infinitesimal morphism deformations. We employ this theory to prove completeness of some Cartan geometries.

This material is based upon works supported by the Science Foundation Ireland under Grant No. MATF634.

2. COFRAMINGS

Lets start with a simpler concept than a Cartan geometry.

2.1. Definitions.

Definition 8. A *coframing* of a manifold M is a 1-form $\omega \in \Omega^1(M) \otimes V$, where V is a vector space of the same dimension as M , so that at each point $m \in M$, the linear map $\omega_m : T_m M \rightarrow V$ is a linear isomorphism. A coframing is also called a *parallelism* or an *absolute parallelism*.

Clearly a coframing is a Cartan geometry modelled on $G/H = V/0$.

Remark 4. There are obvious complex analytic analogues of coframings, which we can call *holomorphic coframings*. Many of the results we prove on coframings have obvious complex analytic analogues.

Definition 9. If two manifolds M_0 and M_1 have coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$ respectively, and $A : V_0 \rightarrow V_1$ is a linear map, then a smooth map $f_1 : X \rightarrow M_1$ is an *A-development* of a smooth map $f_0 : X \rightarrow M_0$ if $f_1^* \omega_1 = A f_0^* \omega_0$. We will also say that f_0 *A-develops* to f_1 , or just say that f_0 develops to f_1 if A is understood. If A is not specified, we will assume that $A = I$ is the identity map.

Definition 10. We will say that *curves A-develop* from a manifold M_0 with a coframing to a manifold M_1 with a coframing if every rectifiable map $\mathbb{R} \rightarrow M_0$ *A-develops* to a map $\mathbb{R} \rightarrow M_1$.

Definition 11. We will say that *complex curves A-develop* from a complex manifold M_0 with a holomorphic coframing to a complex manifold M_1 with a holomorphic coframing if every holomorphic map $f : C \rightarrow M_0$ from a simply connected Riemann surface develops to a holomorphic map to M_1 .

Definition 12. If $\omega \in \Omega^1(M) \otimes V$ is a coframing, and $v \in V$ is a vector, we denote by \vec{v} the unique vector field on M so that $\vec{v} \lrcorner \omega = v$, called a *constant* vector field.

Definition 13. We will say that *lines develop* to a manifold with coframing, or that the coframing is *complete*, or has *complete flow* if each constant vector field complete, i.e. its flow is defined for all time.

Definition 14. We say similarly that *complex lines develop* to a complex manifold with holomorphic coframing, of that the coframing is holomorphically complete, or has *complete holomorphic flow* if each holomorphic constant vector field is holomorphically complete, i.e. its flow is defined for all complex time.

Definition 15. Pick a positive definite inner product on a vector space V . The associated *canonical metric* of a coframing ω is the Riemannian metric for which the $\omega_m : T_m M \rightarrow V$ is an orthogonal linear map for all $m \in M$.

Definition 16. Pick a Hermitian inner product on a complex vector space V . The *canonical Hermitian metric* of a holomorphic coframing is the Hermitian metric for which the $\omega_m : T_m M \rightarrow V$ is unitary for each $m \in M$.

Example 6. Pick a Lie group G . The left (right) invariant Maurer–Cartan 1-form is a left (right) invariant coframing. Lines develop to left translates of 1-parameter subgroups. Curves develop from any Lie group to any other, by solving a finite dimensional Lie equation, as we will see. We will also see that every canonical metric on any Lie group is complete.

Remark 5. The study of completeness of canonical metrics is a small part of the larger picture of Charles Frances [18], in which the completion of the metric space in the canonical metric plays an essential role.

Definition 17. Given a coframing $\omega \in \Omega^1(M) \otimes V$, the *torsion* of ω is the quantity $T : M \rightarrow V \otimes \Lambda^2(V)^*$ given by $d\omega = T\omega \wedge \omega$.

It is easy to prove:

Lemma 2. A coframing is locally isomorphic to the left invariant Maurer–Cartan coframing of a Lie group just when its torsion is constant. A coframing is isomorphic to the left invariant Maurer–Cartan coframing of a Lie group just when lines develop to the coframing and the torsion is constant.

2.2. Flows and coframings.

Remark 6. Let's picture a coframing to which lines develop but curves do not. Take the plane with a coframing of the form

$$\omega = \frac{1}{f} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

for some function $f > 0$. Draw a parabola. Choose the function f to grow rapidly along the parabola, but rapidly approach 1 away from the parabola. Lines develop to this coframing to become lines, but reparameterized to move more quickly as they approach the parabola. As we travel along any line, we eventually move away from the parabola, so eventually go back to a slow steady speed. However, moving along the parabola at unit speed in the framing means at speed f in the usual coframing $\begin{pmatrix} dx \\ dy \end{pmatrix}$, so in some finite time we leave any compact set. As we will see, the canonical metric is incomplete.

Lemma 3 (Clifton [9]). Let M be an n -dimensional manifold with coframing. Every canonical metric of M is complete just when curves develop from \mathbb{R}^n to M .

Proof. Take a coframing $\omega \in \Omega^1(M) \otimes V$ on a manifold M . Take a rectifiable curve $x(t) \in V$. Let $v(t) = x'(t)$. Suppose that the canonical metric is complete. Locally integrate the flow of $\vec{v}(t)$ through any point. In the canonical metric, the flow lines are rectifiable, with locally bounded velocity, so if they are defined on an open time interval, then they extend uniquely to the closure of that interval. The flow line through each point is defined on a maximal open interval, which must therefore be the entire real line.

Take a coframing $\omega \in \Omega^1(M) \otimes V$ on a manifold M . Suppose that curves develop from V to M . Take a geodesic $x(t) \in M$ of a canonical metric on M , parameterized by arc length. Suppose that $x(t)$ does not extend beyond some open interval of time. Then the development to V is

$$y(t) = \int x'(t) \lrcorner \omega dt.$$

Because $x(t)$ is parameterized by arc length, $|dy/dt| = 1$ for all t . This curve $y(t)$ is rectifiable, since dy/dt is bounded and integrable. Extend $y(t)$ to a rectifiable curve on the closure of the open interval on which $x(t)$ is defined, by continuity. Develop $y(t)$ back to M to extend $x(t)$. \square

Example 7. Curves develop to a coframing on a manifold just when they develop to every other coframing which agrees with it outside of some compact set.

Example 8. On the connected sum of two manifolds with coframings, pick a coframing which agrees with the original coframings on the summands away from a compact set where the gluing takes place. Curves develop to the connected sum just when they develop to both summands. The same idea works for developing lines.

Lemma 4. Let M be a manifold with coframing $\omega \in \Omega^1(M) \otimes V$. If a canonical metric on M is complete then, for any integrable function $f : [a, b] \rightarrow V$, the time varying vector field $\vec{f}(t)$ on M is complete.

Proof. Clearly any flow line of $\vec{f}(t)$ has velocity at most $|f(t)|$. Let $x(t)$ be a flow line. The distance of $x(t)$ from x_0 increases at a rate of at most $\int |f(t)| dt$, and therefore for each time t , the integral curve remains in a closed ball. Because the metric is complete, closed balls are compact. Suppose that the flow line is defined for time $0 \leq t < T$. The limit $\lim_{t \rightarrow T} x(t)$ must exist because of the compactness of the ball. Therefore we can extend $x(t)$ continuously to $0 \leq t \leq T$. But then $\dot{x}(t)$ also extends to be locally integrable on $0 \leq t \leq T$, as $\vec{f}(t)$. Therefore $x(t)$ is rectifiable on $0 \leq t \leq T$ and a flow line of $\vec{f}(t)$. But $x(t)$ is defined on a maximal open interval, so must be defined on $[a, b]$. \square

Lemma 5. Let M_1 be a manifold with coframing $\omega_1 \in \Omega^1(M_1) \otimes V_1$. If curves develop from \mathbb{R}^n to M_1 then, for any manifold M_0 with coframing $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and any linear map $A : V_0 \rightarrow V_1$, curves A -develop from M_0 to M_1 .

Proof. Suppose that curves develop from V_1 to M_1 . Then A -develop curves from M_0 to V_1 , and then from V_1 to M_1 . \square

Lemma 6. Take manifolds M_0 and M_1 with coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Take a linear map $A : V_0 \rightarrow V_1$. Let $W_1 \subset V_1$ be the image of A . Curves A -develop from M_0 to M_1 just when, for any integrable function $f : I \rightarrow W_1$ ($I \subset \mathbb{R}$ any interval), the time varying vector field $\vec{f}(t)$ on M_1 is complete.

Proof. Suppose that all of these time varying vector fields $\vec{f}(t)$ are complete. Take any rectifiable curve $m_0(t)$ on M_0 , and let $f(t) = m'_0(t) \lrcorner \omega_0$. Construct the time varying vector field $\vec{A}f(t)$; its flow line is our development.

Conversely, suppose that curves A -develop from M_0 to M_1 . Pick an integrable function $f_1 : I \rightarrow W_1$. We can pick a subspace $W_0 \subset V_0$ complementary to the kernel of A , so that $A : W_0 \rightarrow W_1$ is a linear isomorphism. We can let $f_0 = A^{-1}f_1 : I \rightarrow W_0$. Pick points $m_0 \in M_0$ and $m_1 \in M_1$ and a time $t_0 \in I$. The time-varying vector field $\vec{f}_0(t)$ on M_0 has a flow line $x_0(t)$ through with $x_0(t_0) = m_0$, and the time-varying vector field $\vec{f}_1(t)$ on M_1 has a flow line $x_1(t)$ through with $x_1(t_0) = m_1$. Clearly these are developments of one another, over the time interval during which both curves are defined.

There is a time (possibly infinite) T , $t_0 < T \leq \infty$, when the flow line $x_0(t)$ stops being defined. Lets write this time T as $T = t_0 + \Delta t(m_0, t_0)$. Similarly define $\Delta t(m_1, t_0)$ for any point $m_1 \in M_1$.

Since $x_1(t)$ is a development of $x_0(t)$, where both are defined, we can extend $x_1(t)$ to be defined for as long a time as $x_0(t)$ is defined: extend it to be the development of $x_0(t)$. Then $x_1(t)$ is still a flow line of $\vec{f}_1(t)$. So flow lines on M_1 are defined for at least as long a time interval as those on M_0 : $\Delta t(m_0, t_0) \geq \Delta t(m_1, t_0)$

for any points $m_0 \in M_0$ and $m_1 \in M_1$. In particular, $\inf_{m_1 \in M_1} \Delta t(m_1, t_0) \geq \sup_{m_0 \in M_0} \Delta t(m_0, t_0) > 0$. Therefore $\Delta t(m_1, t_0)$ is bounded from below by a positive function of t_0 .

Pick inner products on V_0 and V_1 so that $A : W_0 \rightarrow W_1$ is an isometry. Equip M_0 and M_1 with the canonical metrics. Fix a point $m_0 \in M_0$. Pick a number $r > 0$ small enough so that the ball of radius r about m_0 (in the canonical metric) is compact. The function

$$\int_{t_0}^{t_1} |f_1(t)| dt$$

is a continuous function of a variable t_1 , where defined. It vanishes at $t_1 = t_0$. Pick $t_1 > t_0$ small enough so that

$$\int_{t_0}^{t_1} |f_1(t)| dt < r.$$

The value of t_1 you choose may depend on the choice of point m_0 and time t_1 .

At time $t_1 > t_0$, the flow line of $\vec{f}_0(t)$ from m_0 and time t_0 stays in the ball of radius at most $\int_{t_0}^{t_1} |f_0| dt$ around m_0 in M_0 . So the flow line stays inside this compact ball of radius r . Therefore the flow line $m_0(t)$ must continue to be defined up to time $t = t_1$. So if

$$\int_{t_0}^{t_1} |f_1(t)| dt \leq r,$$

then

$$\inf_{m_1 \in M_1} \Delta t(m_1, t_0) \geq t_1 - t_0.$$

Divide up the time axis into small increments $t_0, t_1, t_2, \dots, t_N$ so that

$$\int_{t_i}^{t_{i+1}} |f(t)| dt < r$$

on each increment. The flow line $m_1(t)$ of $\vec{f}_1(t)$ will survive for at least enough time to get from each time t_i to the time t_{i+1} , and therefore by induction the flow survives for all time. \square

Summing up:

Proposition 1. On an n -dimensional manifold M with coframing, the following are equivalent:

- (1) some canonical metric is complete,
- (2) every canonical metric is complete,
- (3) curves develop from \mathbb{R}^n ,
- (4) curves I -develop from some manifold with coframing,
- (5) curves A -develop from any manifold with coframing for any A .

Proof. Clearly (1) is equivalent to (2), since the metrics are Lipschitz equivalent. By lemma 3 on page 6, (2) is equivalent to (3). Clearly (3) implies (4) and (5) implies (4). By lemma 5 on the preceding page, (3) implies (5). Suppose that curves I -develop to M from some manifold M_0 with coframing. By lemma 6 on the previous page, (4) implies that bounded time varying vector fields of a suitable form on M are complete, which is independent of choice of manifold M_0 from which we were able to develop curves, and so (4) implies (5). \square

Example 9. In the plane, the coframing

$$\omega = \left(\frac{1}{2 + \sin(x^2 y)} dx \right)$$

clearly has complete canonical metric, since the metric is within bounded dilation of the Euclidean metric. The dual framing is

$$\begin{aligned} \vec{\partial}_1 &= (2 + \sin(x^2 y)) \partial_x, \\ \vec{\partial}_2 &= \partial_y. \end{aligned}$$

Note that $[\vec{\partial}_1, \vec{\partial}_2] = -\cos(x^2 y) x^2 \partial_1$. Along $y = 0$, $[\vec{\partial}_1, \vec{\partial}_2] = -x^2 \partial_x$ is incomplete. Therefore completeness of the canonical metric does not ensure that the brackets of vector fields from a coframing are complete.

Conjecture 1. If the Lie algebra of vector fields generated by a coframing consists entirely of complete vector fields, then the canonical metric of the coframing is complete.

3. IMMERSSED CURVES

Definition 18. A *ordinary vector field* on a manifold M with coframing $\omega \in \Omega^1(M) \otimes V$ is a time varying vector field $\vec{f}(t)$ on M with $f : I \rightarrow V$ bounded, continuous and never vanishing on an interval $I \subset \mathbb{R}$.

Lemma 7. Take two manifolds M_0 and M_1 with coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Take a linear map $A : V_0 \rightarrow V_1$, with image $W_1 \subset V_1$. Immersed curves A -develop from M_0 to M_1 just when all ordinary vector fields $\vec{f}(t)$ on M_1 are complete, for any $f : I \rightarrow W_1$.

Proof. The proof is identical to that of lemma 6 on page 7, but with f nowhere 0. \square

Corollary 1. On an n -dimensional manifold M with coframing, the following are equivalent:

- (1) immersed curves develop from \mathbb{R}^n ,
- (2) immersed curves develop from some n -dimensional manifold with coframing,
- (3) immersed curves A -develop from any manifold with coframing, where A is any injective linear map.

Remark 7. Clifton [9] has an example of a coframing so that lines develop from \mathbb{R}^n but some immersed curves don't.

4. HOLOMORPHIC COFRAMINGS

Lemma 8. Let M_1 be an n -complex-dimensional complex manifold with holomorphic coframing $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Take a linear map $A : V_0 \rightarrow V_1$. If the canonical metric of M_1 is complete then complex curves A -develop from V_0 to M_1 .

Remark 8. It is not known whether there are holomorphic coframings with incomplete canonical metric to which all complex curves develop from a complex vector space.

Proof. Suppose that the canonical metric is complete. Then every real curve in V_0 A -develops to M_1 by lemma 3 on page 6. Suppose that C is a simply connected Riemann surface and that $f : C \rightarrow V_0$ is a holomorphic map. The equations for A -development form a complex analytic system of determined ordinary differential equations of first order, so there is a unique local solution near any point. Take any real curve on C , and develop it. The resulting development extends to a holomorphic development of a neighborhood of the real curve. Because C is simply connected, there is no monodromy, and these local solutions extend to all of C . \square

Lemma 9. Let M_0 and M_1 be complex manifolds with holomorphic coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Let $A : V_0 \rightarrow V_1$ be a linear map. Let W_1 be the image of A . Let U be a simply connected open set in \mathbb{C} . If complex curves A -develop from M_0 to M_1 , then every time varying vector field $\vec{f}(t)$ on M_1 is complete, with $f : U \rightarrow W_1$ any holomorphic function.

Proof. Suppose that complex curves A -develop from M_0 to M_1 . Let $\Delta_r(t)$ be the disk of radius r about any point $t \in \mathbb{C}$. Take a holomorphic function $f_1 : U \rightarrow W_1$. Let $W_0 \subset V_0$ be any complex linear complement to the kernel of A . Let $f_0 = A^{-1}f_1$. For every point $m_0 \in M$ and initial time $t_0 \in U$, there is some radius $r = r(m_0, t_0)$ so that the flow of $\vec{f}_0(t)$ starting at m_0 at time $t = t_0$ is defined for all $t \in \Delta_r(t_0)$, and $0 < r \leq \infty$. Similarly define $r(m_1, t_0)$ for $m_1 \in M_1$.

Since we can develop curves from M_0 , we can develop the flow of $\vec{f}_0(t)$ through any point m_0 into a flow line of $\vec{f}_1(t)$ through any point m_1 . Therefore $r(m_1, t_0) \geq r(m_0, t_0)$. Since this holds for any points $m_0 \in M_0$ and $m_1 \in M_1$,

$$\inf_{m_1 \in M_1} r(m_1, t_0) \geq \sup_{m_0 \in M_0} r(m_0, t_0).$$

Pick some point t' on the boundary of $\Delta_{r(m_1, t_0)}$, so that the flow line $x_1(t)$ does not extend to $t = t'$, and let $t \rightarrow t'$. Then

$$\lim_{t \rightarrow t'} r(x_1(t), t) \rightarrow 0.$$

Therefore

$$(4.1) \quad \lim_{t \rightarrow t'} \sup_{m_0 \in M_0} r(m_0, t) \rightarrow 0.$$

If $t' \in U$ then $\vec{f}_0(t)$ is holomorphic at times t near t' so there is a unique flow line $m_0(t)$ of $\vec{f}_0(t)$ with $m_0(t') = m_0$, defined in some disk of positive radius $r(m_0, t') > 0$. Clearly this contradicts equation 4.1 above. Therefore t' does not belong to U .

Take any real curve $t(s)$ inside U for some real variable $a \leq s \leq b$. Let $x_1(t)$ be the integral curve of $\vec{f}_1(t)$ with $x_1(t(a)) = m_1$. Then $x_1(t)$ is defined for all t in a disk of radius $r(m_1, t(a))$. But then as we vary s , at the first moment $s = s_1$ that $t(s)$ leaves this disk, it doesn't leave U , so we can extend this flow line $m_1(t)$ to values of t near $t(s)$. We can then extend to the disk of radius $r(m_1, t(s_1))$, etc. In this way we produce a sequence of extensions of the flow line to various open intervals, and then extend to their closures, so clearly to all s from $s = a$ to $s = b$.

Therefore the flow line of $\vec{f}_1(t)$ is defined for all t along any real curve inside U . Since U is simply connected, and the local flow lines are unique, there is a unique flow line defined on all of U . \square

Lemma 10. Let M_1 be a complex manifold with holomorphic coframing $\omega_0 \in \Omega^1(M_1) \otimes V$. If complex curves I -develop from *some* holomorphically coframed manifold to M_1 then complex curves develop from V to M_1 .

Proof. Suppose that M_0 is a complex manifold with a holomorphic coframing $\omega_0 \in \Omega^1(M_0) \otimes V$. Suppose that complex curves develop from M_0 to M_1 . Then by lemma 9 on the facing page, all holomorphic time varying vector fields $\vec{f}(t)$ on M are complete, for $f : U \rightarrow V$ a holomorphic function, where $U \subset \mathbb{C}$ is any simply connected open set.

If we take a complex curve

$$f : C \rightarrow \mathbb{C}^n,$$

with C a disk, or $C = \mathbb{C}$, then clearly the developments are the flow lines of $\vec{f}'(t)$. Therefore we only need to consider the problem for $C = \mathbb{CP}^1$, the Riemann sphere. But for \mathbb{CP}^1 , any holomorphic map f is constant, and so the development is just a constant map. \square

Proposition 2. Let M be an n -complex-dimensional complex manifold with holomorphic coframing. Then the following are equivalent:

- (1) complex curves I -develop to M from \mathbb{C}^n ,
- (2) complex curves I -develop to M from *some* n -dimensional complex manifold with holomorphic coframing,
- (3) complex curves A -develop to M from *any* holomorphically framed complex manifold, for any linear map A .
- (4) time varying vector fields $\vec{f}(t)$ on M are complete, when $f(t)$ is any holomorphic function of t , and t lives in some simply connected open set in \mathbb{C} .

Proof. Lemma 9 on the preceding page shows that (2) implies (4). Lemma 10 shows that (2) implies (1). Clearly (1) implies (2) and (3) implies (2). Clearly (1) implies (4), by developing the curve $\int f(t) dt$ from \mathbb{C}^n . If we assume (1), we can A -develop curves from any complex manifold M_0 to \mathbb{C}^n first and then develop to M . Therefore (1) implies (3). To finish, let's see why (4) implies (1). It is clear that for any simply connected complex curve C biholomorphic to the disk or \mathbb{C} , any map $f : C \rightarrow \mathbb{C}^n$ develops to M , by integrating $X_t = \vec{f}'(t)$. But then if C is biholomorphic to \mathbb{CP}^1 , f must be constant, so develops trivially from \mathbb{C}^n to M . \square

5. SYMMETRIES OF COFRAMINGS

Definition 19. A *symmetry* of a coframing ω on a manifold M is a diffeomorphism $F : M \rightarrow M$ so that $F^*\omega = \omega$. An *infinitesimal symmetry* of a coframing ω is a vector field Y so that $0 = \mathcal{L}_Y\omega = 0$.

Proposition 3. If lines develop to a coframing, then any infinitesimal symmetry of that coframing is a complete vector field.

Proof. Suppose that Y is an infinitesimal symmetry of a coframing $\omega \in \Omega^1(M) \otimes V$ on a manifold M . Then $0 = [Y, \vec{v}]$ for each $v \in V$. The flow lines of Y are permuted by the flows of the various \vec{v} . We can therefore take any flow line of Y , say starting at a point m and defined for a time $T > 0$, and flow it around, moving the point m to any other nearby point. Suppose that the constant vector fields \vec{v} of ω are complete. We can move the entire flow line to start at any other nearby point,

and still be defined for time at least T . Therefore all flow lines through all nearby points are defined for at least time T . This means that as we follow any flow line of Y , at each point of that flow line, there is still at least T units of time left to continue flowing. Since $0 < T \leq \infty$, we never run out of time, i.e. the flow lines are defined for all time. \square

Theorem 3 (Kobayashi [23] p. 15, theorem 3.2). The symmetries of a coframing form a finite dimensional Lie group. The Lie algebra of this Lie group is precisely the set of complete infinitesimal symmetries.

Lemma 11. A coframing on a connected manifold is homogeneous just when it has a finite collection of complete vector fields which are local infinitesimal symmetries, and are linearly independent at every point.

Proof. By Kobayashi's theorem, the infinitesimal symmetries form a finite dimensional Lie algebra. This Lie algebra will be spanned by the finite collection of vector fields. By a theorem of Palais [32], a finite dimensional Lie algebra of vector fields, generated by complete vector fields, integrates to a Lie algebra action. The action is locally transitive, because the vector fields are linearly independent. Every orbit is open. The manifold is connected. Therefore the manifold is homogeneous under the group action. \square

6. MORPHISMS OF COFRAMINGS

Definition 20. Suppose that M_0 and M_1 are manifolds with coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Pick a linear map $A : V_0 \rightarrow V_1$. An *A-morphism* is a map $f : M_0 \rightarrow M_1$ so that $f^*\omega_1 = A\omega_0$. We will also refer to an *A-morphism* as just a morphism to avoid reference to a specific map A .

Example 10. Local isomorphisms are precisely *I*-morphisms.

Example 11. The flow lines of the vector fields \vec{v} are morphisms: take any manifold M_1 with coframing $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Pick an element $v_1 \in V_1$. Let $A : \mathbb{R} \rightarrow V_1$, $A(x) = x v_1$. Pick a point $m_1 \in M_1$. Let M_0 be the open interval of \mathbb{R} of times t for which $e^{\vec{v}_1} m_1$ is defined. Let $\omega_0 = dt$. Map $f : M_0 \rightarrow M_1$, $f(t) = e^{t\vec{v}_1} m_1$.

The sense in which development generalizes morphism is obvious.

Proposition 4. Suppose that M_0 and M_1 are manifolds bearing coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Suppose that M_0 is path connected. Take a linear map $A : V_0 \rightarrow V_1$. Pick points $m_0 \in M_0$ and $m_1 \in M_1$. There is an *A-morphism* $f : M_0 \rightarrow M_1$ with $f(m_0) = m_1$ just when every loop in M_0 starting and ending at m_0 develops to a loop in M_1 starting and ending at m_1 .

Proof. Clearly if there is an *A-morphism* $f : M_0 \rightarrow M_1$, then every map $X \rightarrow M_0$ composes with f to produce a development to a map $X \rightarrow M_1$.

On the other hand, suppose that all loops in M_0 , starting and ending at m_0 , develop to loops in M_1 starting and ending at m_1 . It is immediately clear (as in the theory of fundamental groups) that there is a unique smooth map $f : M_0 \rightarrow M_1$ so that $f(m_0) = m_1$ and so that development of paths from M_0 starting at m_0 is carried out by composition with f . It is then clear that f is a morphism, since there is a path on M_0 leaving m_0 with any specified tangent vector at its other end. \square

Lemma 12. Suppose that M_0 and M_1 are manifolds bearing coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Take a linear map $A : V_0 \rightarrow V_1$. A submanifold Z of $M_0 \times M_1$ is locally the graph of a local A -morphism just when

- (1) Z is an integral manifold of the Pfaffian system $\omega_1 = A\omega_0$ and
- (2) ω_0 , pulled back to $M_0 \times M_1$, is a coframing on Z .

Proof. Because the 1-form ω_0 is a coframing on M_0 , its pullback to Z is a coframing just when the composition $Z \subset M_0 \times M_1 \rightarrow M_0$ is a local diffeomorphism. Since the problem is local, we can assume that this composition is a diffeomorphism. So we can assume that Z is the graph of a map $F : M_0 \rightarrow M_1$. Because $\omega_1 = A\omega_0$ on Z , we see that $F^*\omega_1 = A\omega_0$. \square

Lemma 13. If $f : \text{open} \subset M_0 \rightarrow M_1$ is an A -morphism, and $m_0 \in M_0$ and $m_1 = f(m_0)$, then in logarithmic coordinates, $x = \log_{m_0}$ and $y = \log_{m_1}$, the map f is expressed as the linear map $y = Ax$.

Proof. Clearly if there were going to be a morphism $f : U_0 \rightarrow M_1$, it would have to satisfy

$$f(e^{\vec{v}}m_0) = e^{\overrightarrow{Av}}m_1.$$

But this equation determines a map f completely, near m_0 :

$$f(m) = e^{\overrightarrow{A \log_{m_0} m}}m_1.$$

Clearly in logarithmic coordinates, f is the linear map A . \square

Corollary 2. Suppose that $f_n : U_n \text{ open} \subset M_0 \rightarrow M_1$ is a sequence of A -morphisms, say taking points $m_0(n) \in M_0$ to $m_1(n) \in M_1$. If the sequences of points converge, say $m_0(n) \rightarrow m_0$ and $m_1(n) \rightarrow m_1$, then f_n converges with all derivatives to a morphism $f : \text{open} \subset M_0 \rightarrow M_1$ in some neighborhood of m_0 .

Proof. Clearly we can define a map f by making f be A in logarithmic coordinates. By the continuity in Picard's theorem, f is the limit of the various f_n on some open set. Since the various f_n all satisfy $f_n^*\omega_1 = A\omega_0$, so does f . \square

Theorem 4. Suppose that M_0 and M_1 are real analytic manifolds bearing real analytic coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Suppose that M_0 is connected. Take a linear map $A : V_0 \rightarrow V_1$. Suppose that for each $v_0 \in V_0$, the vector field $\overrightarrow{Av_0}$ on M_1 is complete.

Then every A -morphism of coframings $F : U_0 \rightarrow M_1$ defined on a connected open subset $U_0 \subset M_0$ factors uniquely as $F = \tilde{F}\iota$, $\tilde{F} : \tilde{M}_0 \rightarrow M_1$ is an A -morphism of coframings from some covering space of M_0 , and $\iota : U_0 \rightarrow \tilde{M}_0$ is an injective local diffeomorphism covering the inclusion $U_0 \rightarrow M_0$. The covering map $p : \tilde{M}_0 \rightarrow M_0$ yields a map $(p, F) : \tilde{M}_0 \rightarrow M_0 \times M_1$, making \tilde{M}_0 an immersed submanifold of $M_0 \times M_1$.

Proof. Suppose that $M'_0 \subset M_0 \times M_1$ is the graph of an A -morphism on a connected open set of M_0 . Denote by \vec{v}_0 the constant vector fields on M_0 for $v_0 \in V_0$, and by \vec{v}_1 the constant vector fields on M_1 for $v_1 \in V_1$. On $M_0 \times M_1$, the flow of the vector field $\hat{v}_0 = \vec{v}_0 + \overrightarrow{Av_0}$ is tangent to M'_0 for any vector $v_0 \in V_0$. Moreover, each tangent plane of M'_0 is precisely the span of all of these vector fields \hat{v}_0 .

At each point of M'_0 , for any vectors $v_0, w_0 \in V_0$ the bracket $[\hat{v}_0, \hat{w}_0]$ lies tangent to M'_0 , since the vector fields do. So we can express this bracket on M'_0 as a linear

multiple of the various vectors \hat{v}_0 . Take a basis $e_i, i = 1, 2, \dots, m$ for V_0 , and then $\hat{e}_1 \wedge \hat{e}_2 \wedge \dots \wedge \hat{e}_m \wedge [\hat{e}_i, \hat{e}_j] = 0$ for any i and j , at every point of M'_0 .

As we move along the flow of any one of these vector fields \hat{v}_0 , starting at any point of M'_0 , we must find by analyticity that $\hat{e}_1 \wedge \hat{e}_2 \wedge \dots \wedge \hat{e}_m \wedge [\hat{e}_i, \hat{e}_j] = 0$ all along the flow line. We can assume that M'_0 is connected. The tangent spaces of M'_0 are the span of the vector fields \hat{v}_0 . Therefore the flows of these \hat{v}_0 vector fields act locally transitively on M'_0 . But then since M'_0 is connected, these vector fields act transitively on M'_0 .

Let \tilde{M}_0 be the orbit of all of the vector fields \hat{v}_0 through some point of M'_0 . Then \tilde{M}_0 is a submanifold of $M_0 \times M_1$ (perhaps not embedded; see Sussmann [39, 40]), and \tilde{M}_0 contains M'_0 . At every point of \tilde{M}_0 , we find $\hat{e}_1 \wedge \hat{e}_2 \wedge \dots \wedge \hat{e}_m \wedge [\hat{e}_i, \hat{e}_j] = 0$ by analyticity. So \tilde{M}_0 is an integral manifold of the Pfaffian system $\omega_1 = A\omega_0$, invariant under the flows of the various \hat{v}_0 .

Now consider the map $p : \tilde{M}_0 \rightarrow M_0$ given by composing the inclusion $\tilde{M}_0 \subset M_0 \times M_1$ with the forgetful submersion $M_0 \times M_1 \rightarrow M_0$. Clearly $p_*\hat{v}_0 = \vec{v}_0$. Therefore p is a local diffeomorphism. The coframing on \tilde{M}_0 is $p^*\omega_0$.

We need to see why p is a covering map. Pick a point $m_0 \in M_0$, say with logarithm \log_{m_0} a diffeomorphism in a neighborhood U_0 of m_0 . Then clearly logarithmic coordinates around each point in $p^{-1}m_0$ are defined throughout $p^{-1}U_0$. Let $Z = p^{-1}m_0$. Given any point $m'_0 \in M_0$, take a smooth path $x(t)$ with $x(0) = m_0$ and $x(1) = m'_0$ and $x'(t) \neq 0$. Then the time varying vector field $\vec{x}'(t)$ on \tilde{M}_0 will flow the fiber $p^{-1}m_0$ to $p^{-1}m'_0$. Clearly p is a covering map.

If we started off with a morphism defined on an open set $U_0 \subset M_0$, then we can map $U_0 \rightarrow \tilde{M}_0 \subset M_0 \times M_1$, as the graph of that morphism. So this lifts the inclusion $U_0 \rightarrow M_0$ to a local diffeomorphism $U_0 \rightarrow \tilde{M}_0$. \square

Corollary 3. Under the hypotheses of theorem 4, if A is injective, then the map $\tilde{M}_0 \rightarrow M_1$ determined by the A -morphism is an immersion. If moreover M_0 is complete, then every A -morphism $F' : M'_0 \rightarrow M_1$ striking a point in the image of F must factor $F' = \tilde{F}\iota$ where $\iota : \tilde{M}'_0 \rightarrow \tilde{M}_0$ is a local isomorphism from a covering space \tilde{M}'_0 of M'_0 .

Remark 9. If M_0 is itself simply connected and complete, then clearly $\tilde{M}_0 = M_0$, i.e. each morphism extends uniquely to be defined on all of M_0 .

Proof. Completeness of the coframing on M_0 clearly implies completeness of the pullback coframing on any covering space, so on \tilde{M}_0 . Pick manifold M'_0 with coframing $\omega'_0 \in \Omega^1(M'_0) \otimes V_0$. Pick any A -morphism $f' : M'_0 \rightarrow M_1$. In logarithmic coordinates, $y = f'(x') = Ax$ and $y = f(x) = Ax$, so that $x' \mapsto x = x'$ is a local diffeomorphism. Since f and f' are morphisms, they must identify constant vector fields on M'_0 and on M_0 respectively with those on M_1 . Therefore $x' \mapsto x = x'$ is a local isomorphism. By theorem 4 on the previous page, this local isomorphism extends to a local isomorphism of a covering space. \square

Lemma 14. Suppose that $A : V_0 \rightarrow V_1$ is a linear surjection. Suppose that there is an A -morphism of coframings $M_0 \rightarrow M_1$. If M_0 is complete, then M_1 is complete. If every canonical metric of M_0 is complete, then every canonical metric of M_1 is complete.

Proof. Take a splitting $V_0 = V_1 \oplus V_1^\perp$, and then develop lines/curves from V_1 to M_1 by first including these lines/curves into V_0 and then developing into M_0 and then composing with the morphism. \square

Lemma 15. Suppose that $A : V_0 \rightarrow V_1$ is any linear map, with image $W_1 \subset V_1$. Suppose that $q : V_1 \rightarrow V_1/W_1$ is the obvious linear projection. Take a manifold M_1 with coframing ω_1 . The integral manifolds of the Pfaffian system $q\omega_1 = 0$ on M_1 on which ω_1 has rank equal to $\dim W_1$ are locally the images of A -morphisms to M_1 . The image of any A -morphism $M_0 \rightarrow M_1$, with M_0 connected, is an open subset of precisely one of these integral manifolds.

Proof. Without loss of generality, we can assume that as matrices,

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$q = \begin{pmatrix} 0 & I \end{pmatrix},$$

and write out the Pfaffian equations and the result is clear. \square

Lemma 16. Suppose that A is an injective linear map. Any A -morphism of coframings is completely determined by its value at a single point.

Proof. The morphism intertwines the flows of the constant vector fields, and this determines the graph of the morphism locally. \square

Definition 21. Suppose that $\omega \in \Omega^1(M) \otimes V$ is a coframing on a manifold M . Denote the torsion by $T : M \rightarrow V \otimes \Lambda^2(V)^*$, so that $d\omega = T\omega \wedge \omega$.

Remark 10. Suppose that $A : V_0 \rightarrow V_1$ is a linear injection, with image W_1 , and that $q : V_1 \rightarrow V_1/W_1$ is the obvious linear projection. Denote by $\Lambda^2 A : \Lambda^2(V_0) \rightarrow \Lambda^2(V_1)$ the obvious map $\Lambda^2 A(u \wedge v) = A(u) \wedge A(v)$. Suppose that $\omega_1 \in \Omega^1(M_1) \otimes V_1$ is a coframing on a manifold M_1 . The torsion of the Pfaffian system for A -morphisms (i.e. the system $q\omega_1 = 0$) is $qT_1\Lambda^2 A = 0$. Therefore no morphisms map to points where $qT_1\Lambda^2 A \neq 0$.

Let Z be the set of points of M_1 at which $qT_1\Lambda^2 A = 0$. If the framing is real analytic, then Z is a real analytic variety, and stratified by smooth subvarieties. Each stratum is foliated by integral manifolds. If ω_1 has full rank, i.e. rank equal to $\dim V_0$, on the tangent spaces of a leaf on one of these strata, then the leaf has ω_1 as coframing. The inclusion of the leaf into M_1 is a morphism.

Remark 11. Let's consider a noninjective linear map $A : V_0 \rightarrow V_1$. Suppose that M_0 and M_1 are manifolds bearing coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Pick a particular integral manifold of the Pfaffian system $q\omega_1 = 0$. Replace M_1 by this integral manifold. So then we can effectively assume that A is surjective.

If there is an A -morphism, then the fibers of $M_0 \rightarrow M_1$ will be integral manifolds of the Pfaffian system $A\omega_0 = 0$. If we try to construct an A -morphism $M_0 \rightarrow M_1$, we first have to ask whether the Pfaffian system $A\omega_0 = 0$ satisfies the conditions of the Frobenius theorem. If we write $d\omega_0 = T_0\omega_0 \wedge \omega_0$, and we write an exact sequence

$$0 \longrightarrow \ker A \xrightarrow{\iota} V_0 \xrightarrow{A} V_1 \xrightarrow{q} V_1/W_1 \longrightarrow 0,$$

then in order that there could be a morphism $M_0 \rightarrow M_1$, the Frobenius theorem requires $T_0\Lambda^2\iota = 0$ at all points of M_0 . If this holds, then M_0 is foliated by the leaves

of this Pfaffian system. If quotient space $M_{\bar{0}}$ of the foliation is Hausdorff, then it is a manifold with a coframing $\omega_{\bar{0}}$, so that under the projection map $p : M_0 \rightarrow M_{\bar{0}}$, $p^*\omega_{\bar{0}} = q\omega_0$. We then try to construct a \bar{A} -morphism $M_{\bar{0}} \rightarrow M_1$. Note that \bar{A} is injective.

Remark 12. Suppose that A is an injective linear map. Lets consider the local problem of constructing manifolds M_1 which admit A -morphisms from a given manifold M_0 with a given coframing. Without loss of generality we can assume that $A = (I \ 0)$. Take any product $M_0 = M_1 \times U$ with any manifold U with trivial tangent bundle. Take ω_U any coframing on U . Let $\omega_0 = \omega_1 + f\omega_U$ where $f : M_0 \rightarrow V_U^* \otimes (V_1 \oplus V_U)$ is any linear map. Of course, any A -morphism to M_1 will be locally isomorphic to this one for a suitable choice of f .

Corollary 4. Suppose that $A : V_0 \rightarrow V_1$ is an injective linear map with image W_1 . Suppose that $q : V_1 \rightarrow V_1/W_1$ is the obvious linear projection. Take a manifold M_1 with coframing $\omega_1 \in \Omega^1(M_1) \otimes V_1$. Suppose that $d\omega_1 = T_1\omega_1 \wedge \omega_1$. Suppose that $qT_1\Lambda^2 A = 0$.

Then through each point $m_1 \in M_1$, there is a unique maximal immersed submanifold $M_0 = M_0(m_1)$ containing m_1 on which $q\omega_N = 0$, with coframing $\omega_0 = A^{-1}\omega_1$. The inclusion $M_0 \rightarrow M_1$ is an A -morphism.

Every A -morphism $F : M'_0 \rightarrow M_1$, with M'_0 connected and m_1 in the image of F factors as $M'_0 \rightarrow M_0 \rightarrow M_1$, where $M'_0 \rightarrow M_0$ is a local isomorphism of coframings and $M_0 = M_0(m_1)$ for any point m_1 in the image of F .

Proof. Clearly we must take M_0 to be the leaf of the Pfaffian system through a point of M_1 . We can take M_0 to be a maximal leaf. \square

7. THE FROBENIUS–GROMOV THEOREM FOR COFRAMING MORPHISMS

Definition 22. If $\omega \in \Omega^1(M) \otimes V$ is a coframing on a manifold M , then the 1 -torsion of ω is the function $T : M \rightarrow V \otimes \Lambda^2(V)^*$ defined by

$$d\omega = \frac{1}{2}T\omega \wedge \omega.$$

We also write T as $T^{(0)}$. For any integer $p > 1$, the p -torsion is defined recursively $T^{(p)}$ to be the function

$$T^{(p)} : M \rightarrow V \otimes \Lambda^2(V)^* \otimes V^{*\otimes p}$$

given by

$$dT^{(p-1)} = T^{(p)}\omega.$$

Clearly $T = T^{(1)}, T^{(2)}, \dots, T^{(p)}$ at a point $m \in M$ determine the p -jet of T at m .

Lemma 17. Suppose that $\omega \in \Omega^1(M) \otimes V$ is a coframing. Pick any $v_1, v_2, \dots, v_p \in V$. Write out the brackets as

$$\text{ad}(\vec{v}_1) \text{ad}(\vec{v}_2) \dots \text{ad}(\vec{v}_{p-1}) \vec{v}_p = \overrightarrow{\tau_p(v_1, v_2, \dots, v_p)},$$

for some function $\tau_p : M \rightarrow \otimes^p V^* \otimes V$. Then $\tau_1(v) = v$ and inductively

$$\tau_{p+1}(v_1, v_2, \dots, v_{p+1}) = \mathcal{L}_{\vec{v}_1} \tau_p(v_2, v_3, \dots, v_p) - T(v_1 \wedge \tau_p(v_2, v_3, \dots, v_p)).$$

In particular, we can compute the values of $\tau_1, \tau_2, \dots, \tau_p$ at a point $m \in M$ from the values of $T^{(1)}, T^{(2)}, \dots, T^{(p-1)}$ at m .

Proof. Clearly $\tau_1(m)(v) = v$. Moreover, from the definition of T , clearly

$$\tau_2(m)(v_1, v_2) = T(m)(v_1, v_2).$$

Suppose by induction that $\tau_p(m)$ is some function of $T(m), T^{(1)}(m), \dots, T^{(p)}(m)$. Pick some vectors $v_1, v_2, \dots, v_{p+1} \in V$. Let X be the vector field

$$\text{ad}(\vec{v}_2) \text{ad}(\vec{v}_3) \dots \text{ad}(\vec{v}_p) \vec{v}_{p+1}.$$

Then

$$\begin{aligned} \tau_{p+1}(v_1, v_2, \dots, v_{p+1}) &= \omega([\vec{v}_1, X]) \\ &= \mathcal{L}_{\vec{v}_1}(X \lrcorner \omega) - \mathcal{L}_X(\vec{v}_1 \lrcorner \omega) - d\omega(\vec{v}_1, X) \\ &= \mathcal{L}_{\vec{v}_1} \tau_p(v_2, v_3, \dots, v_p) - T(v_1 \wedge \tau_p(v_2, v_3, \dots, v_p)). \end{aligned}$$

□

Corollary 5. Given the value of a coframing ω at a point m , and the value of the j -torsion of the coframing at m , for $j = 1, \dots, p-1$, we can compute the bracket at m : $\text{ad}(\vec{v}_1) \text{ad}(\vec{v}_2) \dots \text{ad}(\vec{v}_{p-1}) \vec{v}_p$ of any collection of up to $p+1$ constant vector fields

Definition 23. Suppose that M_0 and M_1 are manifolds with coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$ and that $A : V_0 \rightarrow V_1$ is a linear map. For any nonnegative integer j , the j -th order obstruction $\text{Ob}^{(j)}$ of the pair (ω_0, ω_1) is the function

$$\text{Ob}^{(j)} = T_1^{(j)}(m_1) = AT_0^{(j)}(m_0) \Lambda^2 A \otimes A^{\otimes(p-1)},$$

on $M_0 \times M_1$, $j = 0, 1, 2, \dots$. The j -th order morphism locus of the pair (ω_0, ω_1) is the set $V_j \subset M_0 \times M_1$ where the obstructions of order not more than j vanish. The morphism locus is the intersection of all of the j -order morphism loci, $j = 0, 1, 2, \dots$.

Definition 24. Suppose that M_0 and M_1 are manifolds with coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$ and that $A : V_0 \rightarrow V_1$ is a linear map. We will say that a point $m_0 \in M_0$ hits a point $m_1 \in M_1$ to order p if (m_0, m_1) lies in the p -th order morphism locus. We will say that a point $m_0 \in M_0$ hits a point $m_1 \in M_1$ if there is a morphism $F : U_0 \rightarrow M_1$ of an open neighborhood U_0 of m_0 so that $F(m_0) = m_1$.

Theorem 5. Suppose that M_0 and M_1 are smooth manifolds with smooth coframings $\omega_0 \in \Omega^1(M_0) \otimes V_0$ and $\omega_1 \in \Omega^1(M_1) \otimes V_1$ and that $A : V_0 \rightarrow V_1$ is a linear map. A point m_0 hits a point m_1 just when (m_0, m_1) belongs to the morphism locus.

Proof. Clearly if a point (m_0, m_1) does not belong to the morphism locus then m_0 does not hit m_1 . So we only need to prove that if (m_0, m_1) belongs to the morphism locus, then m_0 hits m_1 . Let Z be the morphism locus. For each vector $v \in V_0$, let $\bar{v} = \vec{v} + \vec{A}v$, a vector field on $M_0 \times M_1$. These vector fields precisely span the plane field given by $\omega_1 = A\omega_0$. We leave the reader to calculate that

$$\bar{v} \lrcorner d(\text{Ob}^{(j)}(v_0, v_1, w_1, w_2, \dots, w_j)) = \text{Ob}^{(j+1)}(v_0, v_1, w_1, w_2, \dots, w_j, v).$$

It follows (by the Picard uniqueness theorem) that Z is invariant under \bar{v} for any vector $v \in V_0$. So the orbit of the family of all vector fields \bar{v} through each point of Z lies entirely inside Z .

Compute

$$\begin{aligned}\mathcal{L}_{\bar{v}}(\omega_1 - A\omega_0) &= \bar{v} \lrcorner d(\omega_1 - A\omega_0) \\ &= T_1(Av, \omega_1) - AT_0(v, \omega_0).\end{aligned}$$

In particular, at any point of Z , for any vector $w \in V_0$,

$$\bar{w} \lrcorner \mathcal{L}_{\bar{v}}(\omega_1 - A\omega_0) = 0.$$

Therefore at each point of Z , the 1-form $\mathcal{L}_{\bar{v}}(\omega_1 - A\omega_0)$ vanishes on $\omega_1 = A\omega_0$, and is therefore a multiple of $\omega_1 - A\omega_0$. Integrating, we find that the ideal of 1-forms generated by the components of $\omega_1 - A\omega_0$ (in any basis for V_1) is invariant under the flow of \bar{v} through any point of Z . Therefore every vector field \bar{w} , for $w \in V_0$, is carried by the flow of \bar{v} into a linear combination of vector fields of the form \bar{u} for various $u \in V_0$. Differentiating, the bracket $[\bar{v}, \bar{w}]$ must also lie tangent to $\omega_1 = A\omega_0$ through each point of Z .

Indeed we can see this result more directly from lemma 17 on page 16: compute out

$$[\bar{v}, \bar{w}](m_0) = -\overrightarrow{T_0(m_0)(v, w)}(m_0),$$

and

$$[\overrightarrow{Av}, \overrightarrow{Aw}](m_1) = -\overrightarrow{T_1(m_1)(Av, Aw)}(m_0),$$

so that at points of Z ,

$$[\bar{v}, \bar{w}](m_0, m_1) = -\overrightarrow{T_0(m_0)(v, w)}(m_0, m_1).$$

The orbit through any point of $M_0 \times M_1$ of the family of all vector fields \bar{v} for $v \in V_0$ is a smooth submanifold. On the orbit through any point of Z , the Frobenius theorem applies to the Pfaffian system $\omega_1 = A\omega_0$. The orbit is therefore foliated by maximal integral manifolds, whose tangent planes are precisely spanned by the various \bar{v} vector fields. But then the maximal integral manifolds are themselves orbits, and therefore the entire orbit must be a single maximal integral manifold. Its tangent space is precisely the span of the various \bar{v} vector fields. In particular, it projects locally diffeomorphically to M_0 . \square

Example 12. Suppose that M_0 and M_1 are manifolds with coframings

$$\omega_0 \in \Omega^1(M_0) \otimes V_0$$

and

$$\omega_1 \in \Omega^1(M_1) \otimes V_1$$

and that $A : V_0 \rightarrow V_1$ is a linear map. The j -order morphism locus is a real analytic set, and therefore so is the morphism locus. By the descending chain condition, we can cover $M_0 \times M_1$ by open sets so that on each of these open sets, the morphism locus is equal to the j -th order morphism locus for some j . So we can test for local morphisms with purely local calculations at some order.

Corollary 6. Two real analytic coframings agree at a point and have the same torsions of all orders at that point just when they are equal in a neighborhood of that point.

Proof. Let $A = I$. \square

8. HARTOGS EXTENSION OF COFRAMING MORPHISMS

Definition 25. A *Stein manifold* is a complex manifold X which is biholomorphic to a complex submanifold $Y \subset \mathbb{C}^N$, for some integer $N \geq 0$, so that Y is closed as a subset.

Definition 26. An *envelope of holomorphy* of a complex manifold M is a Stein manifold \hat{M} so that

- (1) $M \subset \hat{M}$ is an open subset,
- (2) every holomorphic function on M extends to \hat{M} .

Theorem 6 (Docquier and Grauert [10]). If D is a domain in a Stein manifold, then D has a unique envelope of holomorphy \hat{D} , up to a unique biholomorphism which is the identity on D .

Lemma 18. Suppose that D is a domain in a Stein manifold, with envelope of holomorphy \hat{D} . Then every holomorphic coframing of D extends uniquely to a holomorphic coframing of \hat{D} .

Proof. Take a coframing $\omega \in \Omega^1(D) \otimes V$ on D . Every holomorphic differential form extends uniquely from D to \hat{D} (see McKay [31] p. 10 proposition 2), so extend ω to \hat{D} . If we take a basis for V , we can write

$$\omega = \begin{pmatrix} \omega^1 \\ \omega^2 \\ \vdots \\ \omega^n \end{pmatrix}.$$

The set of points of \hat{D} at which ω fails to be a coframing is the complex hypersurface

$$\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n = 0.$$

Every nonempty complex hypersurface in \hat{D} intersects D (see McKay [31] p. 8 lemma 11). Therefore this complex hypersurface is empty. \square

Theorem 7. Suppose that M is a complex manifold with complete holomorphic coframing. Suppose that D is a domain in a Stein manifold, and D has a holomorphic coframing, and that $f : D \rightarrow M$ is a holomorphic morphism of coframings. Then f extends uniquely to a holomorphic morphism of coframings $f : \hat{D} \rightarrow M$ on the envelope of holomorphy \hat{D} of D .

Proof. The coframing on D extends uniquely to a holomorphic coframing on \hat{D} by lemma 18. The morphism f factors uniquely through a holomorphic morphism on a covering space of \hat{D} , by theorem 4 on page 13, say $f = f' \iota$ with $f' : \hat{D}' \rightarrow M$ a morphism from a covering space, $p : \hat{D}' \rightarrow \hat{D}$, and $\iota' : D \rightarrow \hat{D}'$ a lift of the inclusion $\iota : D \rightarrow \hat{D}$, so $p \iota' = \iota$. On fundamental groups, $p_* \iota' = \iota$. Note that $\iota_* : \pi_1(D) \rightarrow \pi_1(\hat{D})$. It is well known (see McKay [31] p. 7 lemma 6) that ι_* is surjective. Therefore $p_* \iota'_*$ is surjective, and so p_* is surjective. Because p is a covering map, p_* is injective. Therefore p_* is an isomorphism, i.e. $\hat{D} = \hat{D}'$. \square

9. MORPHISMS OF CARTAN GEOMETRIES

We now move on from studying coframings to studying Cartan geometries. We will allow Sharpe's generalization of the concept of Cartan geometry, but we will use our own terminology.

Definition 27 (Sharpe [36] p. 174). A *local model* (H, \mathfrak{g}) is a Lie group H and an H -module \mathfrak{g} , with the structure of a Lie algebra, on which H acts as automorphisms, with an injection $\mathfrak{h} \rightarrow \mathfrak{g}$ of Lie algebras, where \mathfrak{h} is the Lie algebra of H .

Remark 13. Sharpe [36] uses the term “model” for what we call the “local model”. We need the distinction because some of our results require a homogeneous space as model. Keep in mind that a local model might not arise from any homogeneous space, but merely from a Lie subgroup $H \subset G$ which need not be closed. Sharpe's terminology is confusing and its use should be discouraged.

Definition 28. The *local model* of a homogeneous space G/H is (H, \mathfrak{g}) where \mathfrak{g} is the Lie algebra of G .

Remark 14. In all of our examples, the only local models we will consider will be the local models of homogeneous spaces.

Definition 29. Let (H, \mathfrak{g}) be a local model. A (H, \mathfrak{g}) -geometry (also known as a *Cartan geometry* modelled on (H, \mathfrak{g})) on a manifold M is a principal right H -bundle $E \rightarrow M$ and a 1-form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$ called the *Cartan connection*, which satisfies the following conditions:

- (1) Denote the right action of $h \in H$ on $e \in E$ by $r_h e$. The Cartan connection transforms in the adjoint representation:

$$r_h^* \omega = \text{Ad}_h^{-1} \omega.$$

- (2) $\omega_e : T_e E \rightarrow \mathfrak{g}$ is a linear isomorphism at each point $e \in E$.
- (3) For each $A \in \mathfrak{g}$, define a vector field \vec{A} on E (called a *constant vector field*) by the equation $\vec{A} \lrcorner \omega = A$. For $A \in \mathfrak{h}$, the vector fields \vec{A} generate the right H -action:

$$\vec{A}(e) = \left. \frac{d}{dt} r_{e^{tA}} e \right|_{t=0}, \text{ for all } e \in E.$$

Remark 15. We know of no serious example

Definition 30. A *morphism of local models* $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a pair of maps: (1) a Lie group morphism which we denote $\Phi : H_0 \rightarrow H_1$ (and we also denote by Φ the induced morphism of Lie algebras $\Phi : \mathfrak{h}_0 \rightarrow \mathfrak{h}_1$) and (2) a linear map, which we denote $\Phi : \mathfrak{g}_0 \rightarrow \mathfrak{g}_1$, extending the Lie algebra morphism $\Phi : \mathfrak{h}_0 \rightarrow \mathfrak{h}_1$, so that

$$\Phi(h_0 A_0) = \Phi(h_0) \Phi(A_0),$$

for $h_0 \in H_0$ and $A_0 \in \mathfrak{g}_0$.

Definition 31. Suppose that G_0/H_0 and G_1/H_1 are homogeneous spaces. A *morphism* $\Phi : G_0/H_0 \rightarrow G_1/H_1$ of homogeneous spaces is a Lie group morphism $\Phi : G_0 \rightarrow G_1$ which takes H_0 to H_1 .

Example 13. A morphism of homogeneous spaces $\Phi : G_0/H_0 \rightarrow G_1/H_1$ induces the obvious local model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$.

Definition 32. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $E_0 \rightarrow M_0$ is a (H_0, \mathfrak{g}_0) -geometry with Cartan connection ω_0 and that $E_1 \rightarrow M_1$ is a (H_1, \mathfrak{g}_1) -geometry with Cartan connection ω_1 . A *morphism of Cartan geometries* modelled on Φ (also called a Φ -morphism) is an H_0 -equivariant map $F : E_0 \rightarrow E_1$ so that $F^*\omega_1 = \Phi\omega_0$.

Remark 16. Clearly under a morphism F , the flow lines of each vector field \vec{A} on E_0 are carried to the flow lines of $\overrightarrow{\Phi(A)}$. Indeed (in the notation of the previous definition)

$$F'(e_0) \vec{A}(e_0) = \overrightarrow{\Phi(A)}(F(e_0))$$

for every point $e_0 \in E_0$.

Definition 33. If a morphism of Cartan geometries is modelled on a model morphism, and the model morphism is induced from a morphism of homogeneous spaces (as in example 13 on the preceding page), we will say that the morphism of Cartan geometries is modelled on the morphism of homogeneous spaces.

Remark 17. I do not know of a serious example of a Cartan geometry which is modelled on a local model without being modelled on a homogeneous space. Nevertheless, there are serious examples, as we will see, of morphisms of Cartan geometries which are modelled on morphisms of local models, even though the morphisms of local models do not arise from morphisms of homogeneous spaces. It appears that this phenomenon was behind Sharpe's use of what we have called local models, even though Sharpe did not have morphisms specifically described in his work.

Proposition 5. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $\pi_0 : E_0 \rightarrow M_0$ is an (H_0, \mathfrak{g}_0) -geometry. Suppose that $\pi_1 : E_1 \rightarrow M_1$ is an (H_1, \mathfrak{g}_1) -geometry. Suppose that M_0 is connected. Pick a point $e_0 \in E_0$ and a point $e_1 \in E_1$. Let $m_0 = \pi_0(e_0) \in M_0$ and $m_1 = \pi_1(e_1) \in M_1$. There is a morphism $F : E_0 \rightarrow E_1$ modelled on Φ with $F(e_0) = e_1$ if and only if every loop in M_0 starting and ending at m_0 develops (with frames e_0 and e_1) to a loop in M_1 starting and ending at m_1 .

Proof. Suppose that we have a morphism F , so that $F(e_0) = e_1$. Since F is H_0 -equivariant, it induces a unique smooth map $f : M_0 \rightarrow M_1$ by $f \circ \pi_0 = \pi_1 \circ F$. For any map $h : X \rightarrow M_0$ from any manifold X (with boundary, corners, etc.) the map $H : h^*E_0 \rightarrow (f \circ h)^*E_1$ given by $H(m, e) = (f(m), F(e))$ is clearly a Φ -development.

On the other hand, suppose that every loop in M_0 starting and ending at m_0 develops to a loop in M_1 starting and ending at m_1 , with frames e_0 and e_1 . We want to apply our previous results on development of coframings. Pick a smooth loop $\Gamma_0 : [0, 1] \rightarrow E_0$. Then let $\gamma_0 = \pi_0 \circ \Gamma_0$ so $\gamma_0 : [0, 1] \rightarrow M_0$ is a smooth loop. Develop γ_0 to a smooth loop $\gamma_1 : [0, 1] \rightarrow M_1$, with some development $H : \gamma_0^*E_0 \rightarrow \gamma_1^*E_1$ so that $H^*\omega_1 = \Phi\omega_0$. Now let $\Gamma_1 = H \circ \Gamma_0$. Since Γ_0 is a smooth loop, Γ_1 is also a smooth loop. Therefore we can Φ -develop loops to loops for the coframings ω_0 and ω_1 , so by proposition 4 on page 12, there is a unique Φ -coframing morphism $F : E_0 \rightarrow E_1$ so that Φ -coframing development is composition with F . By uniqueness, F is H_0 -equivariant, and therefore is a Cartan geometry morphism. \square

Example 14. If $E \rightarrow M$ is a G/H -geometry, then the bundle map $E \rightarrow M$ is itself a morphism, modelled on the morphism $G \rightarrow G/H$.

Example 15. Suppose that $E \rightarrow M$ is an (H_1, \mathfrak{g}) -geometry, and that $H_0 \subset H_1$ is a closed subgroup. Then $E \rightarrow E/H_0$ is a (H_0, \mathfrak{g}) -geometry (with the same Cartan connection) called a *lift* of $E \rightarrow M$. The obvious map $E/H_0 \rightarrow M$ is a morphism, modelled on the obvious inclusion $(H_0, \mathfrak{g}) \rightarrow (H_1, \mathfrak{g})$. It is intuitively clear what the lift of a morphism should mean.

The lift is “canonical” or “universal” in the following sense.

Lemma 19. Suppose that H_0 is a closed subgroup of H_1 . Suppose that (H_1, \mathfrak{g}) is a local model. Take the obvious inclusion map of local models $\Phi : (H_0, \mathfrak{g}) \rightarrow (H_1, \mathfrak{g})$. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are Cartan geometries modelled on (H_0, \mathfrak{g}) and (H_1, \mathfrak{g}) . Every morphism of Cartan geometries $F : E_0 \rightarrow E_1$ modelled on Φ factors uniquely as $F = F_{\text{lift}} F_0$ where $F_{\text{lift}} : E_1/H_0 \rightarrow E_1/H_1$ is the lift, and F_0 is a uniquely determined local isomorphism.

Proof. Just write F as $F = \text{id} F = F_{\text{lift}} F_0$. \square

Example 16. A *mutation* is a local model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ with $\Phi : H_0 \rightarrow H_1$ an isomorphism of Lie groups and $\Phi : \mathfrak{g}_0 \rightarrow \mathfrak{g}_1$ an H_0 -module isomorphism; see Sharpe [37] p. 154 for examples. A *mutation* of Cartan geometries $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ is a morphism modelled on a mutation and for which the underlying map $M_0 \rightarrow M_1$ is a diffeomorphism.

Definition 34. Say that a model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a *base epimorphism* if $\mathfrak{g}_0/\mathfrak{h}_0 \rightarrow \mathfrak{g}_1/\mathfrak{h}_1$ is onto, etc., and a *fiber epimorphism* if $H_0 \rightarrow H_1$ is onto, etc. and an *epimorphism* if both a base and fiber epimorphism, etc.

We will need some topological lemmas before we can study the global structure of Cartan geometry epimorphisms.

Lemma 20. Suppose that $\pi : E \rightarrow M$ is a fiber bundle mapping and X is a complete vector field on M . Then there is a complete vector field Y on E so that Y lifts X , i.e. $\pi'(e)Y(e) = X(\pi(e))$ for each $e \in E$. Moreover, the set of complete vector fields on E which lift complete vector fields on M acts locally transitively on E .

Proof. Take a locally finite cover of M by relatively compact open sets U_α and a subordinate partition of unity f_α . We can pick the open sets U_α so that above each one, $\pi : E \rightarrow M$ is a trivial bundle, say $F_\alpha : \pi^{-1}U_\alpha \rightarrow U_\alpha \times Q$, where Q is a typical fiber. Any lift Y_α of $f_\alpha X$ will have the form

$$Y_\alpha(m, q) = (f_\alpha(m) X(m), Z_\alpha(m, q)),$$

for some $Z(m, q) \in T_q Q$. We can pick Z_α to have compact support in $U_\alpha \times Q$. Let $Y = \sum_\alpha Y_\alpha$, which is well defined because the partition of unity is locally finite.

Suppose that the flow of Y through some point e_0 exists only for a time T . Let $m_0 = \pi(e_0)$. The flow of X exists for all time, and $e^{tX} m_0$ belongs to a finite number of open sets U_α , let's say to U_1, U_2, \dots, U_N . So for time t near T , $e^{tX} m_0$ stays inside $U_1 \cap U_2 \cap \dots \cap U_N$. Expressed in terms of the trivialization over U_1 , we will find that inside $\pi^{-1}(U_1 \cap U_2 \cap \dots \cap U_N)$, the vector field Y has the form

$$Y(m, q) = \left(X(m), \sum_{j=1}^N Z_j(m, q) \right),$$

for a finite set of compactly supported functions Z_j . Therefore the flow of m is $\dot{m} = X(m)$, extending for time past T , and the flow of q is $\dot{q} = \sum_{j=1}^N Z_j(e^{tX} m_0, q)$,

which has compact support in q , so is complete. Clearly we can alter the choice of Z_1 as we like to ensure that Y has required value at a single point, so that the tangent space of E at each point is spanned by complete lifts. \square

Corollary 7. The composition of fiber bundle maps is a fiber bundle map.

Proof. Suppose that $P \rightarrow Q$ and $Q \rightarrow R$ are fiber bundle maps. Take the set V_R of complete vector fields on R . Take the set V_Q of complete lifts of those vector fields on Q . Take the set V_P of complete lifts of those vector fields on P . Each set of vector fields acts transitively, and lifts the previous, so each vector field in V_P lifts a vector field in V_R . By a theorem of Ehresmann [15] or McKay [30], $P \rightarrow R$ is a fiber bundle mapping. \square

Theorem 8. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are Cartan geometries, and that $E_0 \rightarrow E_1$ is a morphism of Cartan geometries, modelled on a model epimorphism. Suppose that M_0 is complete, and that M_1 is connected. Then M_1 is complete and $E_0 \rightarrow E_1$ is a fiber bundle map.

Remark 18. We conjecture that $M_0 \rightarrow M_1$ is a fiber bundle map.

Proof. Suppose that $F : E_0 \rightarrow E_1$ is a morphism modelled on an epimorphism. Flow lines are carried to flow lines, so the constant flow on E_1 is complete on the image of F . Therefore the image of F is an H_1 -invariant union of path components of E_1 , and so is all of E_1 : F is a surjective submersion and E_1 has complete flow. Because the map $E_0 \rightarrow E_1$ takes the complete vector field \vec{A} to $\overrightarrow{\Phi(A)}$, the map $E_0 \rightarrow E_1$ is a fiber bundle map; see Ehresmann [15] or McKay [30]. \square

Lemma 21. Pick Cartan geometries $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ with local models (H_0, \mathfrak{g}_0) and (H_1, \mathfrak{g}_1) . Suppose that M_0 is connected. Pick a local model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$.

A Φ -morphism $E_0 \rightarrow E_1$, if one exists, is completely determined by its value at a single point, i.e. if $F_0, F_1 : E_0 \rightarrow E_1$ are two morphisms, which agree at a single point of E_0 , then $F_0 = F_1$ everywhere on E_0 .

Remark 19. So, roughly speaking, the space of morphisms is of dimension at most the dimension of \mathfrak{g}_1 .

Proof. On the graph of a morphism, for each $A \in \mathfrak{g}_0$, the flow of the vector field \vec{A} on E_0 intertwines with the flow of $\overrightarrow{\Phi(A)}$ on E_1 . Therefore on $E_0 \times E_1$, the graph of a morphism is acted on locally transitively by these vector fields. This determines the morphism in a maximal connected open subset of E_0 . Then H_0 -equivariance determines the morphism at all points of E_0 . \square

Lemma 22. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are Cartan geometries with those local models and with Cartan connections ω_0 and ω_1 . A submanifold Z of $E_0 \times E_1$ is locally the graph of a local morphism just when

- (1) Z is an integral manifold of the Pfaffian system $\omega_1 = \Phi\omega_0$ and
- (2) the obvious map $E_0 \times E_1 \rightarrow E_0$ restricts to a local diffeomorphism $Z \rightarrow E_0$.

Proof. Since the problem is local, we can assume that $Z \subset E_0 \times E_1 \rightarrow E_0$ is a diffeomorphism to an open set $Z_0 \subset E_0$. So we can assume that Z is the graph of a map $F : E_0 \rightarrow E_1$. Because $\omega_1 = \Phi\omega_0$ on Z , we see that $F^*\omega_1 = \Phi\omega_0$. Therefore

$F_*\vec{A} = \vec{A}$. Because the infinitesimal generators of the right H_0 -action on E_0 are the vector fields \vec{A} for $A \in \mathfrak{h}_0$, F is equivariant under the identity component of H_0 .

Since our problem is local, we can assume that E_0 and E_1 are trivial, $E_0 = M_0 \times H_0$ and $E_1 = M_1 \times H_1$. Denote the identity component of H_0 as H_0^0 . We can assume that Z is contained inside $M_0 \times H_0^0 \times M_1 \times H_1$. Therefore we can extend the map F by H_0 -equivariance to be defined on E_0 . \square

Definition 35. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are Cartan geometries with those local models. The Φ -obstruction between E_0 and E_1 is the function $K_1\Lambda^2\Phi - \Phi K_0 : E_0 \times E_1 \rightarrow \Lambda^2(\mathfrak{g}_0/\mathfrak{h}_0) \otimes \mathfrak{g}_1$. Let $q : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1/\Phi\mathfrak{g}_0$ be the obvious quotient map. The Φ -obstruction of E_1 is the function $qK_1\Lambda^2\Phi$.

Corollary 8. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are Cartan geometries with those local models and with Cartan connections ω_0 and ω_1 . Suppose that the Φ -obstruction vanishes at every point of $E_0 \times E_1$. Consider the Pfaffian system $\omega_1 = \Phi\omega_0$. Through each point of $E_0 \times E_1$ there passes a unique maximal connected integral manifold Z of this Pfaffian system. Let E'_0 be the union of H_0 -orbits of the points of Z . Then H_0 acts freely and properly on E'_0 . Let $M'_0 = E'_0/H_0$. The composition $E'_0 \rightarrow E_0 \times E_1 \rightarrow E_0$ is an H_0 -equivariant local diffeomorphism, quotienting to a local diffeomorphism $M'_0 \rightarrow M_0$. The 1-form ω_0 pulls back to Cartan connection on $E'_0 \rightarrow M'_0$, and the map $E'_0 \rightarrow E_0$ is a local isomorphism of Cartan geometries. The composition $E'_0 \rightarrow E_0 \times E_1 \rightarrow E_1$ is a Φ -morphism. Every Φ -morphism $F : U_0 \rightarrow E_1$ from an open subset $U_0 \subset E_0$ whose graph intersects E'_0 factors through $E'_0 \rightarrow E_1$.

Proof. By lemma 22 on the previous page, we can see locally that E'_0 is just going to be the graph of a local isomorphism. Factoring of morphisms through $E'_0 \rightarrow E_1$ is clear from lemma 21 on the preceding page. \square

Theorem 9. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are real analytic Cartan geometries with those local models. Suppose that M_1 is complete.

Every real analytic Φ -morphism from a connected open subset $U_0 \subset E_0$ to E_1 extends uniquely to a real analytic Φ -morphism $\tilde{E}_0 \rightarrow E_1$ from the pullback Cartan geometry $\tilde{E}_0 = p^*E_0$ of some covering map $p : \tilde{M}_0 \rightarrow M_0$ with H_0 -invariant lift

$$\begin{array}{ccc} U_0 & \xrightarrow{\quad} & \tilde{E}_0 \\ & \searrow & \downarrow \\ & & E_0. \end{array}$$

Remark 20. Ehresmann [14] claimed a special case of this theorem, without proof. Kobayashi [22] also claimed without proof the same special case, as a consequence of his incorrect main theorem (which also appeared without proof).

Proof. Suppose that the bundles of the Cartan geometries are $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$, with Cartan connections ω_0 and ω_1 . By hypothesis, we have a local morphism on some open sets. By theorem 4 on page 13, we can extend this local morphism to a morphism of coframings $F : \tilde{E}_0 \rightarrow E_1$ on some covering space \tilde{E}_0 of E_0 . Moreover, $\tilde{E}_0 \subset E_0 \times E_1$ is a submanifold (by construction in theorem 4 on page 13), a union

of integral manifolds of the Pfaffian system $\omega_1 = A\omega_0$. Since the construction is H_0 -invariant, the resulting coframing will transform in the adjoint H_0 -representation on \tilde{E}_0 . So the H_0 -action on \tilde{E}_0 is free and proper, because it is on $E_0 \times E_1$. Let $\tilde{M}_0 = \tilde{E}_0/H_0$. Clearly \tilde{M}_0 is a smooth manifold and $\tilde{E}_0 \rightarrow \tilde{M}_0$ is a principal right H_0 -bundle. Pull ω_0 back to $E_0 \times E_1$, and then restrict to \tilde{E}_0 to produce a Cartan connection. Map $\tilde{E}_0 \rightarrow E_1$ by restricting the forgetful map $E_0 \times E_1 \rightarrow E_1$. This map is a covering map, and H_0 -equivariant, and therefore descends to a covering map $\tilde{M}_0 \rightarrow M_0$. \square

9.1. The Frobenius–Gromov theorem for morphisms of Cartan geometries.

Definition 36. Suppose that $E \rightarrow M$ is a Cartan geometry with local model (H, \mathfrak{g}) and Cartan connection ω . Any function $f : M \rightarrow W$ to any vector space W has differential $df = \nabla f \omega$, say, where $\nabla f : E \rightarrow W \otimes \mathfrak{g}^*$. Clearly we can define $\nabla^2 f = \nabla \nabla f$, and so on to all orders,

$$\nabla^j f : E \rightarrow W \otimes \mathfrak{g}^{*\otimes j}.$$

An easy calculation yields:

Lemma 23. Suppose that $E \rightarrow M$ is a Cartan geometry with local model (H, \mathfrak{g}) . Suppose that W is an H -module, and f is a smooth section of $E \times_H W$. Suppose that $\rho : H \rightarrow \mathrm{GL}(W)$ is the H -representation on W . In other words, $f : E \rightarrow W$ and

$$r_h^* f = \rho(h)^{-1} f.$$

Then under right H -action,

$$r_h^* \nabla^j f = \rho(h)^{-1} \otimes (\mathrm{Ad}(h)^{-1})^{\otimes j}.$$

Definition 37. The curvature of the Cartan geometry is given by is a function

$$K : E \rightarrow \mathfrak{g} \otimes \Lambda^2(\mathfrak{g}/\mathfrak{h})^*$$

given by

$$d\omega + \frac{1}{2}[\omega, \omega] = K\omega \wedge \omega.$$

So we have defined $\nabla^0 K = K, \nabla^1 K, \nabla^2 K, \dots$ and we see that the functions $\nabla^j K$ for $j < p$ determine the p -jet of K . Clearly

$$\nabla^j K : E \rightarrow \mathfrak{g} \otimes \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \bigotimes^j \mathfrak{g}^*$$

Definition 38. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $\pi_0 : E_0 \rightarrow M_0$ and $\pi_1 : E_1 \rightarrow M_1$ are real analytic manifolds bearing real analytic Cartan geometries with those local models.

Say that a point $e_0 \in E_0$ Φ -hits a point $e_1 \in E_1$ to order p if

$$\nabla^j K_1(e_1) \Lambda^2 \Phi \otimes \bigotimes^j \Phi = \Phi \nabla^j K_0(e_0),$$

for $j = 0, 1, 2, \dots, p$.

Say that a point $e_0 \in E_0$ Φ -hits a point $e_1 \in E_1$ if there is an open set $U \subset M$ containing the point $m_0 = \pi_0(e_0)$ and a morphism $F : \pi_0^{-1}U \rightarrow E_0$ so that $F(e_0) = e_1$. We will say *hits* instead of Φ -hits if Φ is understood.

Theorem 10. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are real analytic manifolds bearing real analytic Cartan geometries with those local models.

For each compact set $K \subset M_0 \times M_1$, there is an integer p so that if a point $e_0 \in E_0$ hits a point $e_1 \in E_1$ to order p and if $(\pi_0(e_0), \pi_1(e_1)) \in K$ then e_0 hits e_1 .

In particular, the set of points $(e_0, e_1) \in E_0 \times E_1$ so that e_0 hits e_1 is an analytic variety.

Proof. Clearly it is enough to prove the result locally, so we can assume that $E_0 = M_0 \times H_0$ and $E_1 = M_1 \times H_1$. Fix a compact subset $K_0 \subset H_0$ and a compact subset $K_1 \subset H_1$, each containing a neighborhood of the origin. To each compact set $K \subset M_0 \times M_1$, associate the set K' of points (m_0, h_0, m_1, h_1) with $(m_0, m_1) \in K$ and $h_0 \in K_0$ and $h_1 \in K_1$.

We now have two notions of hitting: for the Cartan geometry and for the coframings given by the Cartan connections. By lemma 22 on page 23, any coframing Φ -morphism on an open subset of E_0 is locally a Cartan geometry Φ -morphism. So we can hit with the coframings just when we can hit with the Cartan geometries.

Clearly by theorem 5 on page 17, we can find an integer p so that the points hit to order p lying inside K' are the points hit. By right invariance of the p -hitting and hitting criteria, the same value of p will work everywhere: any point e_0 that hits a point e_1 to p -th order lies in the domain of a (locally unique) coframing Φ -morphism F taking e_0 to e_1 . \square

Remark 21. In the notation of the previous theorem, the set of pairs $(e_0, e_1) \in E_0 \times E_1$ such that e_0 hits e_1 is an analytic variety, invariant under the action

$$(e_0, e_1) h_0 = (e_0 h_0, e_1 \Phi(h_0)).$$

Therefore it projects to an analytic variety in the quotient $E_0 \times_{H_0} E_1$.

In the special case where $M_1 = G_1/H_1$ is the homogeneous space with its standard flat geometry, this analytic variety is G_1 -invariant too, so actually quotients to M_0 : i.e. the set of points of M_0 that hit one and hence every point in the model G_1/H_1 forms an analytic subvariety of M_0 . But this is just the set of points of M_0 near which there is a Φ -morphism to G_1/H_1 . In particular, if the morphism $\Phi : G_0/H_0 \rightarrow G_1/H_1$ is an immersion, then taking $M_1 = G_1/H_1$, we find that then either M_0 is everywhere locally hitting G_0/H_0 or there are no points of M_0 near which there is a local Φ -morphism to G_1/H_1 .

In particular, if $\Phi = I$ is the identity, we recover the fact that local homogeneity on a dense set implies local homogeneity everywhere.

Proposition 6. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that M_0 and M_1 are real analytic manifolds bearing real analytic Cartan geometries with those models. Suppose that M_1 has constant curvature. Then the set of points of M_0 which hit some point of M_1 (or equivalently hit a given point of M_1 , or equivalently hit every point of M_1) is a union of path components of M_0 .

Proof. We can assume that M_1 has zero curvature, by replacing its local model by a mutation. If a Φ -morphism is defined near some point $m_0 \in M_0$, say taking m_0 to some point $m_1 \in G_1/H_1$, then it is defined at all points nearby, say on some open set $U_0 \subset M_0$. Its various translates under local isomorphisms of M_1 form a family of Φ -morphisms so that we can get any point of U_0 to map to any point of M_1 by some translate. Therefore the set of points of M_0 hitting any given point m_1

is open and not empty. So above each point $m_0 \in U_0$, there is some point $e_0 \in E_0$ at which

$$0 = \Phi \nabla^j K_0(e_0),$$

for $j = 0, 1, \dots, p$. This equation is H_0 -invariant, and therefore holds on an open H_0 -invariant subset of E_0 . But then by analytic continuation it holds everywhere above a union of path components of M_0 . \square

Proposition 7. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is an epimorphism of local models. Suppose that M_0 and M_1 are real analytic manifolds bearing real analytic Cartan geometries with those models. Suppose that M_0 has constant curvature. Then the set of points of M_1 which are hit by some point of M_0 (or equivalently are hit by a given point of M_0 , or equivalently are hit by every point of M_0) is a union of path components of M_1 .

Proof. We can assume that M_0 has zero curvature, by replacing its model by a mutation. If a Φ -epimorphism hits some point $m_1 \in M_1$, then it hits all points nearby, say on some open set $U_1 \subset M_1$. Its various translates under local isomorphisms of M_0 form a family of Φ -morphisms so that we can get any point of M_0 to map to any point of U_1 by some translate. Therefore the set of points of M_1 hit by a given point m_0 is open and not empty. So above each point $m_1 \in U_1$, there is some point $e_1 \in E_1$ at which

$$0 = \nabla^j K_1(e_1) \wedge^2 \Phi \otimes \bigotimes^j \Phi,$$

for $j = 0, 1, \dots, p$. This equation is H_1 -invariant, and therefore holds on an open H_1 -invariant subset of E_1 . But then by analytic continuation it holds everywhere above a union of path components of M_1 . \square

10. STRATIFICATION THEOREM FOR MORPHISMS OF CARTAN GEOMETRIES

We recall the stratification theorem for algebraic group actions:

Theorem 11 (Rosenlicht [34]). Pick a field k . If an algebraic group G over k acts algebraically on an algebraic variety V over k , then there is a dense G -invariant k -open subset $U \subset V$ so that U/G is a algebraic variety over k , $U \rightarrow U/G$ a regular morphism, and $k(U)^G = k(U/G)$.

Corollary 9. If a real algebraic group G acts algebraically on a real algebraic variety V , then there is a collection of G -invariant algebraic subvarieties $V = V_0 \supset V_1 \supset V_2 \dots$, of successively lower dimensions, so that, if we let $U_j = V_j \setminus V_{j+1}$, then

- (1) U_j/G is a smooth algebraic variety over k , and
- (2) the map $U_j \rightarrow U_j/G$ is a smooth submersion for each j and
- (3) the function fields are $\mathbb{R}(U)^G = \mathbb{R}(U/G)$ and
- (4) each variety U_j/G is affine.

Proof. We pick an open set $U_0 \subset V_0 = V$ as in theorem 11, and set $V_1 = V_0 \setminus U_0$. But then take the points where the map $U_0 \rightarrow U_0/G$ is not a submersion, and add them to V_1 , and thereby remove them from U_0 . Repeat inductively. If U_1/G is not affine, then cut out a proper subvariety from it, say $W_1 \subset U_1/G$, so that $U_1/G \setminus W_1$ is affine. Then let W'_1 be the preimage of W_1 in U_1 , and let $U'_1 = U_1 \setminus W'_1$ and $V_2 = V_1 \cup W'_1$, etc. to inductively arrange that all of the quotient varieties are affine. \square

Recall that a rational map $f : X \rightarrow Y$ from a real analytic space X to an affine variety Y is a map defined away from some analytic subvariety so that the composition with a rational function is rational, i.e. locally a quotient of real analytic functions.

Definition 39. Suppose that V is a real analytic variety and X is a real algebraic variety. A *striating map* is map $I : V \rightarrow X$ (not necessarily continuous) with a choice of closed subvarieties $V = V_0 \supset V_1 \supset V_2 \dots$ (called the *strata*) of successively lower dimension and closed affine subvarieties $X_0, X_1, X_2 \dots \subset X$ so that

- (1) the varieties X_1, X_2, \dots are affine and
- (2) if we let $U_j = V_j \setminus V_{j+1}$, then U_j is smooth and
- (3) $I|_{U_j} : U_j \rightarrow X$ is a smooth map of locally constant rank and
- (4) $I|_{U_j} : U_j \rightarrow X$ has image inside $X_j \subset X$ and
- (5) $I|_{U_j} : V_j \rightarrow X_j$ is a rational map for each j .

Remark 22. A striating map endows each set U_j with a smooth analytic foliation: the level sets of I . So V has a smooth foliation (the level sets of I) except on a subvariety V_1 , but V_1 itself has a smooth foliation (again the level sets of I) except on a subvariety V_2 , etc.

Remark 23. We will consider the closed subvarieties V_j and X_j to be part of the data of the striating map.

Example 17. By Rosenlicht's theorem, if a real algebraic group G acts algebraically on a real algebraic variety V , then there is a G -invariant striating map. On each stratum, this striating map is algebraic and submersive. We can take X to be just the formal disjoint union of the various U_j/G , so that each U_j/G is a closed and open subset of X . It is not clear if the hypotheses of Rosenlicht's theorem ensure that there is one striating map through which all others factor.

Example 18. Suppose that $I : V \rightarrow X$ is any regular map of real algebraic varieties. We can then let V_1 be the set of points where V is not smooth, together with those points where I is not of maximal rank. Restrict I to V_1 and let V_2 be the set of points of V_1 where V_1 is singular or I is not of maximal rank, etc. We set $U_j = V_j \setminus V_{j+1}$. In this way I is canonically a striating map. Clearly not every striating map comes about this way.

Lemma 24. If $I : M \rightarrow X$ is a striating map, and G is a real analytic Lie group with real analytic action on M with smooth quotient M/G , and if I is G -invariant, then I descends to a striating map $I : M/G \rightarrow X$.

Proof. The strata of I are G -invariant so descend to analytic subvarieties, as does the map I . Descent doesn't alter rank of invariant maps. \square

Definition 40. A *refinement* $I' : V \rightarrow X$ of a striating map $I : V \rightarrow X$ is a striating map for which $I' = I$ at every point of V , but whose strata V'_j are strictly larger: $V_j \subset V'_j$.

Lemma 25. Suppose that $I : V \rightarrow X$ is a striating map invariant under a real analytic group action of a group G on a real analytic variety V and $F : W \rightarrow V$ is a G -equivariant real analytic map between real analytic varieties with real analytic G -action. Then there is a refinement $J : W \rightarrow X$ of $I \circ F$ which is a G -invariant striating map.

Proof. For the moment, replace X by the algebraic Zariski closure of the image of $I \circ F$, and V by the preimage of that algebraic Zariski closure under I . So we can assume that F has image striking U_0 . Therefore $I \circ F : F^{-1}U_0 \rightarrow X_0$ is well defined. Then take the set of points of $F^{-1}U_0$ where $I \circ F$ has rank locally constant, say W_0 , as our first stratum. Repeat inductively on the complement of W_0 . Note that the complement of W_0 might have some components mapping under $I \circ F$ into X_0 , so we might have to attach a copy of X_0 to X_1 . This is consistent with the definition of striating map. \square

Definition 41. A local model (H, \mathfrak{g}) is *algebraic* if H is a linear algebraic group and the representation of H on \mathfrak{g} is algebraic.

Definition 42. A Cartan geometry is of *algebraic type* if its local model is algebraic.

Remark 24. A Cartan geometry of algebraic type need not be in any sense algebraic. For example, all pseudo-Riemannian geometries are of algebraic type.

Theorem 12. Suppose that M_0 and M_1 are real analytic manifolds bearing real analytic Cartan geometries. Suppose that the Cartan geometry on M_1 is of algebraic type. Then the set of pairs $(m_0, m_1) \in M_0 \times M_1$ so that m_0 hits m_1 (for a given local model morphism) is an analytic subvariety of $M_0 \times M_1$.

Suppose that $W_0 \subset M_0$ and $W_1 \subset M_1$ are relatively compact open sets. Then there is a pair of striating maps $I_0 : W_0 \rightarrow X$ and $I_1 : W_1 \rightarrow X$, so that a point $m_0 \in W_0$ hits a point $m_1 \in W_1$ just when $I_0(m_0) = I_1(m_1)$. So a point of W_0 hits a point of W_1 just when the points lie on corresponding leaves on corresponding strata. These striating maps can be chosen to be invariant under the pseudogroups of local isomorphisms of the Cartan geometries.

Proof. The problem of determining whether m_0 hits m_1 is the same problem as asking if the quantity

$$\nabla^j K_1(e_1) \Lambda^2 \Phi \otimes \bigotimes^j \Phi$$

can be made equal to the quantity

$$\Phi \nabla^j K_0(e_0),$$

by $H_0 \times H_1$ -action, for $j = 1, 2, \dots, p$ at some points e_0 and e_1 over points m_0 and m_1 respectively, for some sufficiently large integer p . This integer p will depend on the choice of sets W_0 and W_1 .

Let $\kappa_1^j = \nabla^j K_1(e_1) \Lambda^2 \Phi \otimes \bigotimes^j \Phi$. Let $\kappa_0^j = \Phi \nabla^j K_0(e_0)$. Let

$$J_0 = (\kappa_0^0, \kappa_0^1, \dots, \kappa_0^p)$$

and

$$J_1 = (\kappa_1^0, \kappa_1^1, \dots, \kappa_1^p)$$

So J_1 is a function on E_1 valued in some H_1 -module, say V , and is H_1 -equivariant. We must determine if the H_1 -orbit in V of J_1 is the same as the H_1 -orbit in V of J_0 .

By corollary 9 on page 27, there is an H_1 -invariant striating map $I : V \rightarrow X$, which is a submersion on each stratum, which distinguishes orbits of the H_1 -action on V . Therefore $I_1 = I \circ J_1 : E_1 \rightarrow X$ is an H_1 -invariant map. But it might not be striating, because its rank could change at various points. However, it is close to what we need: while it might not be striating, its level sets are precisely the points

of E_1 at which J_1 takes on values in a given H_1 -orbit. So our next problem is to refine the maps I_1 and $I_0 = I \circ J_0$ to become striating maps, i.e. locally constant rank on each stratum.

By lemma 25 on page 28, we can assume that I_0 is a striating map by refinement, and the same for I_1 . The strata of X could be altered in each of these processes of refinement, but (as we see in the proof of lemma 25 on page 28) only by moving various strata X_i into various other lower strata X_{i+1} repeatedly. So we can assume without loss of generality that I_0 and I_1 have the same strata in X . Moreover, we can ensure that $I_0 = I_1$ precisely at points at which $I \circ J_0 = I \circ J_1$.

So now both I_0 and I_1 are striating, and H_1 -invariant, so descend to striating maps $I_0 : M_0 \rightarrow X$ and $I_1 : M_1 \rightarrow X$ respectively. Moreover, these maps have level sets given precisely by the points of W_0 or W_1 above which J_0 or J_1 reach particular H_1 -orbits. Therefore a point $m_0 \in M_0$ hits a point $m_1 \in M_1$ just when $I_0(m_0) = I_1(m_1)$. \square

Remark 25. In the previous theorem, if H_1 is compact, then its space of invariants is a suitable choice of X and I_0 and I_1 can be chosen to be the quotient mapping to X ; see Procesi [33] p. 556, theorem 2. In particular, both I_0 and I_1 will then be smooth real analytic maps to affine space, with image inside the affine algebraic variety X .

Corollary 10. Suppose that M_0 and M_1 are real analytic manifolds bearing real analytic Cartan geometries with those models. Suppose that M_1 is of algebraic type. Pick a point $m_0 \in M_0$. Let Z be the set of points $m_1 \in M_1$ so that m_0 hits m_1 . Then Z is dense if and only if Z is Zariski dense.

Remark 26. For simplicity, we stated the results in this section for real manifolds and varieties, but their complex analytic analogues will have obvious formulations and identical proofs.

Corollary 11 (Dumitrescu [12]). Suppose that M_0 and M_1 are complex manifolds bearing holomorphic Cartan geometries. Suppose that M_1 is of algebraic type.

- (1) Suppose that M_0 admits no nonconstant meromorphic functions. Pick a point $m_1 \in M_1$. If some point of M_0 hits m_1 , then all points of M_0 hit m_1 .
- (2) Suppose instead that M_1 admits no nonconstant meromorphic functions. Pick a point $m_0 \in M_0$. If some point of M_1 is hit by m_0 , then all points of M_1 are hit by m_0 .

Proof. Since there are no nonconstant meromorphic functions on M_0 (or M_1), the striating map I_0 (or I_1) must be constant. \square

Example 19. Suppose that M_1 is a real analytic manifold with a real analytic projective connection. The set of points of M_1 through which there are flat totally geodesic surfaces is a closed real analytic subvariety of M_1 . Indeed we let $M_0 = \mathbb{P}^2$ imbedded as a linear subspace in \mathbb{P}^n , let G_0 be the subgroup of $\mathbb{P}\mathrm{SL}(n+1, \mathbb{R})$ preserving M_0 , and let H_0 be the subgroup of G_0 fixing a point.

Example 20. Complexifying the last example: suppose that M_1 is a complex manifold with holomorphic projective connection. Suppose that M_1 has no nonconstant meromorphic functions. Then either every point of M_1 lies on a flat totally geodesic complex surface, or no point does.

11. AVATARS: MORPHISMS COVERING THE IDENTITY MAP

Remark 27. It is unclear how many different morphisms $E_0 \rightarrow E_1$ between two fixed Cartan geometries $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ could induce the same underlying map $M_0 \rightarrow M_1$.

We want a weaker notion than isomorphism, allowing the Cartan geometry to change on a fixed manifold.

Definition 43. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism, and that the induced linear map $\Phi : \mathfrak{g}_0/\mathfrak{h}_0 \rightarrow \mathfrak{g}_1/\mathfrak{h}_1$ is an isomorphism of H_0 -modules. We will call Φ an *avatar*.

In our earlier terminology, an avatar is a *base isomorphism*.

Theorem 13. Pick an avatar $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$. Pick a (H_0, \mathfrak{g}_0) -geometry $E_0 \rightarrow M$. Let

$$E_1 = E_0 \times_{H_0} H_1.$$

Let ω_1 be the 1-form on $E_0 \times H_1$ which at a point (e_0, h_1) is

$$\omega_1 = h_1^{-1} dh_1 + \text{Ad}(h_1)^{-1} \Phi \omega_0.$$

Then ω_1 is the pullback of a unique 1-form, which we also denote by ω_1 , on E_1 . Moreover $E_1 \rightarrow M$ is a (H_1, \mathfrak{g}_1) -geometry. Define $F_{\text{avatar}} : E_0 \rightarrow E_1$ given by composition of $e_0 \in E_0 \mapsto (e_0, 1) \in E_0 \times H_1$ with $E_0 \times H_1 \rightarrow E_1$. This map F_{avatar} is a Φ -morphism, called the Φ -*avatar* of E_0 .

Lets write $\bar{\omega}_0$ to denote the soldering form $\omega_0 + \mathfrak{h}_0 \in \Omega^1(E_0) \otimes \mathfrak{g}_0/\mathfrak{h}_0$. Then the curvature of the avatar is

$$K_1 \bar{\omega}_1 \wedge \bar{\omega}_1 = \text{Ad}(h_1)^{-1} \left(K_0 \bar{\omega}_0 \wedge \bar{\omega}_0 + \frac{1}{2} [\Phi \omega_0, \Phi \omega_0] - \Phi [\omega_0, \omega_0] \right).$$

Every Φ -morphism F of Cartan geometries factors uniquely as $F = F_1 F_{\text{avatar}}$ where F_1 is a uniquely determined local isomorphism of (H_1, \mathfrak{g}_1) -geometries.

Remark 28. Our terminology is a bit vague, as to whether Φ or F is the avatar, but it should be clear that there is essentially a unique Cartan avatar F_{avatar} modelled on a given model avatar.

Proof. The right action of H_0 on $E_0 \times H_1$ is

$$(e_0, h_1) h_0 = (e_0 h_0, \Phi(h_0)^{-1} h_1), h_0 \in H_0.$$

This action commutes with the right H_1 -action

$$(e_0, h_1) k_1 = (e_0 h_0, h_1 k_1), k_1 \in H_1.$$

The 1-form ω_1 is invariant under the right H_0 -action, and transforms in the adjoint representation under the right H_1 -action. Moreover ω_1 vanishes on the orbits of the H_0 -action, and therefore descends to a 1-form on E_1 . Clearly ω_1 is a Cartan connection on E_1 . The curvature is a simple calculation.

Define $F_{\text{avatar}} : E_0 \rightarrow E_1$ given by composition of $e_0 \in E_0 \mapsto (e_0, 1) \in E_0 \times H_1$ with $E_0 \times H_1 \rightarrow E_1$. This map $F = F_{\text{avatar}}$ is equivariant for the H_0 -action, and $F^* \omega_1 = \Phi \omega_0$, so a Φ -morphism.

Suppose that $F : E_0 \rightarrow E'_1$ is a Φ -morphism, where $E'_1 \rightarrow M'_1$ is a G_1/H_1 -geometry. Define a map (which we also denote by F) say $F : E_0 \times H_1 \rightarrow E'_1$, by $F(e_0, h_1) = F(e_0) h_1$. This map is clearly a submersion to E'_1 ; its fibers are

precisely the right H_0 -orbits. This map is invariant under the right H_0 -action, so descends to a local diffeomorphism $F_1 : E_1 \rightarrow E'_1$. This local diffeomorphism is H_1 -equivariant.

Pulling back the Cartan connection ω'_1 on E'_1 to E_0 , because $E_0 \rightarrow E'_1$ is a morphism,

$$F^* \omega'_1 = \Phi \omega_0.$$

But then pulling back to $E_0 \times H_1$,

$$F^* \omega'_1 = h_1^{-1} dh_1 + \text{Ad}(h_1)^{-1} \Phi \omega_0 = \omega_1.$$

Therefore the map $F_1 : E_1 \rightarrow E'_1$ is a local isomorphism of G_1/H_1 -geometries. \square

Example 21 (From affine connections to projective connections). Let

$$G_0 = \text{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n, \quad H_0 = \text{GL}(n, \mathbb{R}),$$

so G_0/H_0 is affine n -space. Write elements of G_0 as matrices

$$g_0 = \begin{pmatrix} 1 & 0 \\ x_0 & h_0 \end{pmatrix}$$

and elements of H_0 as

$$\begin{pmatrix} 1 & 0 \\ 0 & h_0 \end{pmatrix}.$$

Let $G_1 = \mathbb{P}\text{SL}(n+1, \mathbb{R})$, let H_1 be the subgroup fixing the point $[0, 0, \dots, 0, 1] \in \mathbb{RP}^n$. Let Φ be the obvious composition $G_0 \rightarrow \text{GL}(n+1, \mathbb{R}) \rightarrow \mathbb{P}\text{SL}(n+1, \mathbb{R})$; then Φ is an avatar of homogeneous spaces. This avatar takes a manifold with affine connection to the same manifold with the induced projective connection. This is *not* necessarily the normal projective connection. By theorem 13 on the previous page, there is a unique morphism associating a projective connection to an affine connection.

Example 22 (From Riemannian metrics to affine connections). Let $G_0 = \text{O}(n) \rtimes \mathbb{R}^n$ and $H_0 = \text{O}(n)$. Let $G_1 = \text{GL}(n, \mathbb{C}) \rtimes \mathbb{R}^n$ and $H_1 = \text{GL}(n, \mathbb{C})$. The obvious map $\Phi : G_0/H_0 \rightarrow G_1/H_1$ is an avatar, which takes a Riemannian manifold and forgets the Riemannian metric, remembering only the Levi-Civita connection.

Example 23 (Induced effective geometries). Suppose that $N \subset H$ is a closed subgroup which is a normal subgroup of G . We have an obvious avatar $\Phi : (H, \mathfrak{g}) \rightarrow (H/N, \mathfrak{g}/\mathfrak{n})$. By theorem 13 on the preceding page, there is a unique avatar between any G/H -geometry and some unique $(G/N)/(H/N)$ -geometry on the same manifold. In particular, if we take N to be the largest normal subgroup of G contained in H , then the resulting avatar could be called the *induced effective Cartan geometry*; see Sharpe [37] for the definition of an effective Cartan geometry.

Example 24 (Fefferman constructions). If $G_0 \subset G_1$ and H_0 contains $G_0 \cap H_1 \subset H_0$, and the composition $\mathfrak{g}_0 \rightarrow \mathfrak{g}_1 \rightarrow \mathfrak{g}_1/\mathfrak{h}_1$ is surjective, then the associated morphism of homogeneous spaces is called a *Fefferman* morphism of homogeneous spaces. A morphism of Cartan geometries modelled on a Fefferman morphism is called a *Fefferman construction*; see Čap [5]. By lemma 19 on page 22 and theorem 13, there is a unique Fefferman construction with a given model: the composition of the lift modelled on $G_0/(G_0 \cap H_1) \rightarrow G_0/H_0$ with the obvious avatar modelled on $G_0/(G_0 \cap H_1) \rightarrow G_1/H_1$.

Remark 29. Let's say that a local model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is *target natural* if to every manifold with (H_0, \mathfrak{g}_0) -geometry there is associated a Φ -morphism to a manifold with (H_1, \mathfrak{g}_1) -geometry, and this association is invariant under local isomorphisms. Clearly we have examples of target natural morphisms from Riemannian geometry and projective connections above. Our theorem above says that avatars of homogeneous space geometries are target natural. They are the only known target natural morphisms, besides the construction of a normal projective connection from any projective connection, and the construction of a torsion-free connection from any affine connection.

Let's say that a local model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is *source natural* if to every manifold with (H_1, \mathfrak{g}_1) -geometry, there is associated a manifold with (H_0, \mathfrak{g}_0) -geometry, and a Φ -morphism between these manifolds, and this association is invariant under local isomorphisms. The lift of a Cartan geometry is modelled on a source natural morphism. The classification of source and target natural morphisms of local models is not known or even conjectured.

11.1. Avatars and reductive structure group.

Definition 44. Let's say that a local model (H, \mathfrak{g}) is *nearly reductive* if \mathfrak{g} splits as an H -module into a sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$.

Remark 30. Sharpe [37] calls a local model *reductive* under this condition. Sharpe's terminology is almost certain to confuse the reader into thinking that H (or perhaps G) is supposed to be a reductive linear algebraic group. Sharpe admits (p. 197 footnote 17) that the terms could be confused in this way.

Definition 45. If $\pi : E \rightarrow M$ is a Cartan geometry modelled on a nearly reductive model (H, \mathfrak{g}) , then the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ splits the Cartan connection ω into a sum $\omega = \omega^H + \omega^\perp$ with ω^H a 1-form valued in \mathfrak{h} and ω^\perp a 1-form valued in V .

Example 25. The splitting of the Cartan connection is an avatar, modelled on the model avatar

$$\Phi : (H, \mathfrak{g}) \rightarrow (H, \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}).$$

The 1-form ω^H is a connection for the bundle $E \rightarrow M$.

12. MORPHISMS AS INTEGRAL MANIFOLDS OF A PFAFFIAN SYSTEM

Definition 46. If $E \rightarrow M$ is a Cartan geometry with Cartan connection ω , with model (H, \mathfrak{g}) , then the *curvature* of the Cartan geometry is the function $K : E \rightarrow \mathfrak{g} \otimes \Lambda^2(\mathfrak{g}/\mathfrak{h})^*$ for which

$$d\omega + \frac{1}{2}[\omega, \omega] = \frac{1}{2}K\omega \wedge \omega.$$

(In this definition, $\omega \wedge \omega$ is the form valued in $\Lambda^2(\mathfrak{g}/\mathfrak{h})$ given by $\omega \wedge \omega(v, w) = \omega(v) \wedge \omega(w)$.)

Three natural problems arise in studying morphisms of Cartan geometries:

- (1) How many morphisms are there from a fixed Cartan geometry to a fixed Cartan geometry?
- (2) How many morphisms are there from a fixed Cartan geometry to all possible Cartan geometries?
- (3) How many morphisms are there from all possible Cartan geometries to a fixed Cartan geometry?

Each of these vague questions leads to a different Pfaffian system, whose integral manifolds are locally the graphs of morphisms of the required form.

Example 26. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are Cartan geometries with these models, with Cartan connections ω_0 and ω_1 respectively. On $E_0 \times E_1$, consider the Pfaffian system $\omega_1 = \Phi\omega_0$. We denote by $\Lambda^2\Phi : \Lambda^2(\mathfrak{g}_0/\mathfrak{h}_0) \rightarrow \Lambda^2(\mathfrak{g}_1/\mathfrak{h}_1)$ the linear map $\Lambda^2\Phi(v \wedge w) = F(v) \wedge F(w)$. The torsion of the Pfaffian system at any integral element is $K_1\Lambda^2\Phi - \Phi K_0$.

Let $V \subset E_0 \times E_1$ be the subset of points on which $K_1\Lambda^2\Phi = \Phi K_0$. Suppose that $V^{\text{sm}} \subset V$ is some open subset which is locally a submanifold of $E_0 \times E_1$. Then the Pfaffian system restricts to V^{sm} to satisfy the conditions of the Frobenius theorem. Therefore if V^{sm} is not empty, then it is foliated by integral manifolds of the Pfaffian system. The 1-form ω_0 restricts to a coframing on one of these integral manifolds just when that integral manifold is locally the graph of a local morphism. Clearly the integral manifolds living inside V^{sm} therefore form a finite dimensional family. Indeed if the local models arise from homogeneous space models, then this family is never larger in dimension than the associated family of morphisms of the models.

Keep in mind that there might be singular points of V through which integral manifolds pass. If the Cartan geometry is real analytic, then V is stratified into smooth submanifolds, so that a Zariski open subset of each integral manifold will have to lie inside one of the strata, and so we could in principle find all integral manifolds in this way. In the real analytic category, we can therefore justify the claim that the local morphisms from open subsets of M_0 to open subsets of M_1 form a finite dimensional family of no larger dimension than the associated family of morphisms of the models.

For example, if the homogeneous space G_0/H_0 is one dimensional, then $\Lambda^2\Phi = 0$, and $K_0 = 0$, so there is no torsion and the Pfaffian system foliates E_1 by integral manifolds, which are locally the graphs of local morphisms from open sets of G_0/H_0 .

Example 27. Instead of trying to construct morphisms between specific Cartan geometries, we could ask if there are any morphisms from any Cartan geometries to a fixed Cartan geometry. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $\Phi : \mathfrak{g}_0 \rightarrow \mathfrak{g}_1$ is an injective linear map. Suppose that $E_1 \rightarrow M_1$ is a (H_1, \mathfrak{g}_1) -geometry. Let $q : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1/\Phi(\mathfrak{g}_0)$ be the obvious quotient map. On E_1 , consider the Pfaffian system $q\omega_1 = 0$. The torsion of this Pfaffian system is

$$qK_1\Lambda^2\Phi$$

Again we let V be the set of points on which the torsion vanishes, and let V^{sm} be any open subset of V which is locally a submanifold of E_1 . Again, on V^{sm} the Pfaffian system satisfies the conditions of the Frobenius theorem, so is foliated by integral manifolds. Integral manifolds of the Pfaffian system on which ω_1 is a coframing (valued in $\mathfrak{g}_0!$) are locally graphs of local morphisms from submanifolds with (H_0, \mathfrak{g}_0) -geometry.

Similar remarks to those of the previous example apply concerning integral manifolds through singular points of V .

13. MORPHISMS AND COMPLETE FLOWS

Definition 47. A local model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is called *immersive downstairs* if the induced linear map $\Phi : \mathfrak{g}_0/\mathfrak{h}_0 \rightarrow \mathfrak{g}_1/\mathfrak{h}_1$ is injective.

Proposition 8. Pick a homogeneous space morphism

$$\Phi : G_0/H_0 \rightarrow G_1/H_1$$

which is immersive downstairs. Pick a (H_1, \mathfrak{g}_1) -geometry $E_1 \rightarrow M_1$. Let

$$q : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1/\Phi(\mathfrak{g}_0)$$

be the obvious linear quotient map. Suppose that

- (1) $qK_1\Lambda^2\Phi = 0$ and
- (2) $\mathcal{L}_{\overrightarrow{\Phi(A)}}K_1\Lambda^2\Phi = 0$ for each $A \in \mathfrak{g}_0$ and
- (3) $\overrightarrow{\Phi(A)}$ is a complete vector field, for each $A \in \mathfrak{g}_0$.

Then each point of E_1 lies in the image of a morphism from a mutation G'_0/H_0 of a covering of G_0/H_0 . This morphism is unique up to G'_0 -action. In particular, if $K_1\Lambda^2\Phi = \Phi[\cdot, \cdot]_{\mathfrak{g}_0}$, we can take G'_0 to be a covering of G_0 .

Proof. From example 27, we can see that (1) ensures that E_1 is foliated by images of Φ -morphisms. Next, (2) ensures that each of the leaves is a Cartan geometry with constant curvature. Any Cartan geometry satisfies

$$[\vec{A}, \vec{B}] = \overrightarrow{[A, B]} + \overrightarrow{K(A, B)},$$

so that a Cartan geometry with constant curvature is a mutation for the bracket

$$[A, B]' = [A, B] + K(A, B).$$

Condition (3) ensures completeness, so that the geometry is the image of a local isomorphism from its model. The model must have the same structure group H_0 , being only a mutation. We can replace the model by a covering space to arrange that the model is a covering of a mutation of G_0/H_0 . \square

Remark 31. The image of an immersive downstairs morphism is an immersed submanifold. If it is complete and flat, then it is a quotient of its model G_0/H_0 , up to replacing the model by a covering space G'_0/H_0 . The quotient is by a group action of a discrete subgroup $\Gamma \subset G'_0$.

14. HARTOGS EXTENSION OF CARTAN GEOMETRY MORPHISMS

Lemma 26 (Matsushima and Morimoto [28]). A complex Lie group is Stein if and only if the identity component of its center contains no complex torus.

Proposition 9. Pick

- (1) a holomorphic local model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ and
- (2) a holomorphic (H_0, \mathfrak{g}_0) -geometry $E_0 \rightarrow M_0$ and
- (3) a complete holomorphic (H_1, \mathfrak{g}_1) -geometry $E_1 \rightarrow M_1$ and
- (4) a Φ -morphism $F : E_0 \rightarrow E_1$.

Suppose that H_1 is Stein and M_0 is a domain in a Stein manifold with envelope of holomorphy \hat{M}_0 .

Then the holomorphic principal H_0 -bundle $E_0 \rightarrow M_0$ extends to a holomorphic principal H_0 -bundle $E'_0 \rightarrow \hat{M}_0$ if and only if the Cartan geometry on E_0 extends

to a holomorphic Cartan geometry $E'_0 \rightarrow \hat{M}_0$. If either extension exists, then both are unique and F extends to a unique Φ -morphism $F : E'_0 \rightarrow E_1$.

Proof. The Cartan geometry $E_0 \rightarrow M_0$ extends to a Cartan geometry $E'_0 \rightarrow \hat{M}_0$ just when the holomorphic principal H_0 -bundle $E_0 \rightarrow M_0$ extends to a holomorphic principal H_0 -bundle $E'_0 \rightarrow \hat{M}_0$; see McKay [31] p. 19 theorem 8. By theorem 7 on page 19, the morphism F extends to $F : E'_0 \rightarrow E_1$. \square

15. MORPHISM DEFORMATIONS

Definition 48. A family of Cartan geometries modelled on a local model (H_0, \mathfrak{g}_0) is a choice of

- (1) a principal H_0 -bundle $\pi : E \rightarrow M$ and
- (2) a foliation of M , i.e. a bracket-closed vector subbundle $VM \subset TM$ (and we denote by VE the preimage of VM by π'),
- (3) a section ω_0 of the vector bundle $V^*E \otimes \mathfrak{g}_0 \rightarrow E$ so that
- (4) for each leaf $M_0 \subset M$ of VM , the bundle $\pi^{-1}M_0 \rightarrow M_0$ is a Cartan geometry with ω_0 as Cartan connection.

Definition 49. If the foliation VM of a family of Cartan geometries is the kernel of a smooth submersion to a manifold, say S , we will call S the *parameter space* of the family.

Remark 32. It is often convenient to assume that the parameter space has a chosen point, say $s_0 \in S$, so that we imagine that the deformation is deforming a specific Cartan geometry $E_{p_0} \rightarrow M_{p_0}$.

Definition 50. Pick a local model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$. A Φ -morphism deformation is a choice of

- (1) deformation of Cartan geometries $E \rightarrow M$ modelled on (H_0, \mathfrak{g}_0) , with notation as in definition 48, and
- (2) a Cartan geometry $E_1 \rightarrow M_1$ modelled on (H_1, \mathfrak{g}_1) and
- (3) an H_0 -equivariant map $F : E \rightarrow E_1$ so that
- (4) above each leaf of the foliation of M , F is a Φ -morphism of Cartan geometries.

Definition 51. Suppose that $F_0 : E_0 \rightarrow E_1$ is a morphism of Cartan geometries. A morphism deformation $F : E \rightarrow E_1$ is a morphism deformation of F_0 if there is a map $g : E_0 \rightarrow E$ which immerses E_0 as a maximal integral manifold of VE , and so that $F \circ g = F_0$.

16. UNIVERSAL MORPHISM DEFORMATIONS

Definition 52. A deformation $F : E \rightarrow E_1$ of a morphism $E_0 \rightarrow E_1$ is *versal* along E_0 if for every deformation $F' : E' \rightarrow E_1$ of E_0 , there is an I -morphism $G : E' \rightarrow E$ so that $F' = F \circ G$.

Definition 53. A versal deformation is *universal* if all of its induced I -morphisms to all other versal deformations are immersions.

Lemma 27. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a immersive local model morphism, i.e. an immersion on H_0 and on \mathfrak{g}_0 . Let $q : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1/\Phi\mathfrak{g}_0$ be the obvious linear quotient map. Suppose that $E_1 \rightarrow M_1$ is a (H_1, \mathfrak{g}_1) -geometry. Suppose that

the Φ -obstruction of E_1 vanishes. Then there is a unique universal Φ -deformation $F : E \rightarrow E_1$ from a principal H_0 -bundle $\pi : E \rightarrow M$ up to isomorphism. The universal Φ -deformation has $E = E_1$, $F = \text{Id}$, $M = E/H_0$, $VE = (q\omega_1 = 0)$, $\omega = \omega_1$, and $VM = \pi'VE$.

Proof. First let's see that $\pi : E \rightarrow M$ is a morphism deformation, with morphism $F = \text{Id}$. Because $E_1 \rightarrow M_1$ is a principal right H_1 -bundle, and $H_0 \subset H_1$ is a subgroup, H_0 acts freely and properly on $E = E_1$. Therefore the quotient $M = E/H_0$ is a manifold, and the projection $\pi : E \rightarrow M$ is a principal right H_0 -bundle. (Indeed $\pi : E \rightarrow M$ is just the lift of the H_1 -geometry to an H_0 -geometry.) The vector bundle VE is an H_0 -invariant subbundle of TE , and bracket closed, hence consists entirely of its own Cauchy characteristics, so descends to a vector subbundle $VM \subset TM$ and $VE = (\pi')^{-1}VM$. Moreover, VM is bracket closed because VE is. The rest is clear.

Next, let's see why this morphism deformation is versal. Pick $F' : E' \rightarrow E_1$ any morphism deformation. Let $G = F' : E' \rightarrow E_1$, viewed as an I -morphism deformation. Clearly we find $F' = F \circ G$.

Next, let's see why this morphism deformation is universal. Suppose that $F' : E' \rightarrow E_1$ is another versal Φ -morphism deformation. Then there is an I -morphism $G : E_1 \rightarrow E'$ so that $F = F' \circ G$. But $F = \text{Id}$, so $\text{Id} = F' \circ G$. Therefore G is an immersion.

Uniqueness of the universal morphism deformation is clear. \square

Example 28. Suppose that $G_1 = \mathbb{P}\text{SL}(n+1, \mathbb{R})$ and H_1 is the subgroup fixing a point $p_0 \in \mathbb{RP}^n$. Let $G_0 \subset G_1$ be the subgroup fixing a projective line in \mathbb{RP}^n containing p_0 , and let $H_0 = G_0 \cap H_1$. Let $\Phi : G_0 \rightarrow G_1$ be the obvious inclusion mapping. Then our lemma says that every projective connection (i.e. G_1/H_1 -geometry) has a universal space of pointed geodesics M , and a universal deformation of geodesics. It has a smooth parameter space of geodesics if and only if the Φ -obstruction Pfaffian system consists in the leaves of a continuous map.

Remark 33. It is not clear how to build a universal deformation of a nonimmersive morphism.

Example 29. It is well known that every scalar ordinary differential equation of some order, say order $n+1$, induces a Cartan geometry on its configuration space. To be more precise, write $\text{Sym}^n(\mathbb{R}^2)^*$ for the space of homogeneous polynomials of degree n in two variables x and y , write $C_n \subset \text{GL}(2, \mathbb{R})$ for the group $\pm I$ if n is even and I if n is odd, and let $G_1 = (\text{GL}(2, \mathbb{R})/C_n) \rtimes \text{Sym}^n(\mathbb{R}^2)^*$.

Let $\mathcal{O}(n)$ be the set of all pairs (L, q) so that $L \subset \mathbb{R}^2$ is a line through the origin and $q \in \text{Sym}^n(L)^*$ is a homogeneous polynomial on L of degree n . There is an obvious map $\mathcal{O}(n) \rightarrow \mathbb{RP}^1$ given by $(L, q) \mapsto L$. The manifold $\mathcal{O}(n)$ is the total space of the usual line bundle also called $\mathcal{O}(n)$ over \mathbb{RP}^1 .

Let G_1 act on $\mathcal{O}(n)$ by

$$(g, p)(L, q) = (gL, q \circ g^{-1} + p|_L).$$

Let H_1 be the subgroup preserving the point $(L_0, 0)$ where L_0 is the horizontal axis in \mathbb{R}^2 .

It is well known (see Lagrange [25, 26], Fels [16, 17], Dunajski and Tod [13], Godliński and Nurowski [21], Doubrov [11]) that every scalar ordinary differential

equation of order $n + 1$,

$$\frac{d^{n+1}y}{dx^{n+1}} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right)$$

imposes a G_1/H_1 -geometry on the surface parameterized by the variables x, y , and that this G_1/H_1 -geometry is invariant under fiber-preserving transformations of the ordinary differential equation. Therefore we can view any G_1/H_1 -geometry on a surface as a natural geometric generalization of an ordinary differential equation of order $n + 1$.

Let $G_0 = \mathrm{GL}(2, \mathbb{R})/C_n$ and $H_0 = G_0 \cap H_1$. So $G_0/H_0 = \mathbb{RP}^1$. The orbits of G_0 in $\mathcal{O}(n)$ are precisely the graphs of the global sections of $\mathcal{O}(n)$ given by taking a homogeneous polynomial $Q(x, y)$ of degree n on \mathbb{R}^2 , and mapping $L \in \mathbb{RP}^1 \mapsto (L, Q|_L) \in \mathcal{O}(n)$. In particular, we have an obvious morphism $\Phi : G_0/H_0 \rightarrow G_1/H_1$ of homogeneous spaces: the zero section $\mathbb{RP}^1 \rightarrow \mathcal{O}(n)$ of the bundle map $\mathcal{O}(n) \rightarrow \mathbb{RP}^1$.

If we have a G_1/H_1 -geometry $E \rightarrow M$ which is constructed out of an ordinary differential equation of degree $n + 1$, then the Φ -morphisms of open sets of \mathbb{RP}^1 are precisely the local solutions, equipped with a natural projective connection. The universal morphism deformation is just the family of all solutions of the original scalar ordinary differential equation. Therefore in all scalar ODE systems of degree at least 2 are examples of universal morphism deformations of Cartan geometries.

Definition 54. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a immersive local model morphism, i.e. an immersion on H_0 and on \mathfrak{g}_0 . Let $q : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1/\Phi\mathfrak{g}_0$ be the obvious linear quotient map. Suppose that $E_1 \rightarrow M_1$ is a G_1/H_1 -geometry.

We will say that the G_1/H_1 -geometry $E_1 \rightarrow M_1$ is Φ -tame (or just *tame* if Φ is understood) if

- (1) the Φ -obstruction of E_1 vanishes (so that in particular $E_1 \rightarrow M_1$ has a universal morphism deformation)
- (2) the universal morphism deformation $E \rightarrow M$ of $E_1 \rightarrow M_1$ admits a parameter space S , and
- (3) the map $E \rightarrow S$ is a smooth fiber bundle morphism with compact fibers.

17. THE EQUATION OF FIRST ORDER DEFORMATION

Definition 55. Pick a local model monomorphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$. Let $q : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1/\Phi\mathfrak{g}_0$ be the obvious linear projection map.

Suppose that $F : E \rightarrow E_1$ is a Φ -morphism deformation, with bundles $H_0 \rightarrow E \rightarrow M$ and $H_1 \rightarrow E_1 \rightarrow M_1$. Clearly $qF^*\omega_1$, a section of the vector bundle $(TE/VE)^* \otimes (\mathfrak{g}_1/\Phi\mathfrak{g}_0)$. Since $qF^*\omega_1$ is clearly H_0 -equivariant, it is also a section of the vector bundle $Q_E = (TE/VE)^* \otimes_{H_0} (\mathfrak{g}_1/\Phi\mathfrak{g}_0) \rightarrow M$; we will write δF to mean $qF^*\omega_1$ viewed as a section of the vector bundle $Q_E \rightarrow M$. This object δF will be called the *first order deformation* of F . The restriction of δF to a leaf of VM will be called the first order deformation of that leaf.

Definition 56. In the notation of the previous definition, for any vector $A \in \mathfrak{g}_1$, write \bar{A} for $A + \mathfrak{h}_1 \in \mathfrak{g}_1/\mathfrak{h}_1$.

Definition 57. Suppose that $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a representation, and that $E \rightarrow M$ is a (H, \mathfrak{g}) -geometry or a family of (H, \mathfrak{g}) -geometries. The *covariant derivative* ∇ is

the operator on sections of $E \times_H V$ given by

$$\nabla s = ds + \rho(\omega)s.$$

Lemma 28. Consider a local model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$. Consider a morphism deformation, in the notation of definition 50, modelled on Φ . Suppose that the parameter space S is an interval $S \subset \mathbb{R}$, and let $t : S \rightarrow \mathbb{R}$ be the inclusion mapping. The first order deformation is $\delta F = a dt$ where $a : E \rightarrow \mathfrak{g}_1/\Phi\mathfrak{g}_0$ is a uniquely determined H_0 -equivariant function, i.e. a section of $E \times_{H_0} (\mathfrak{g}_1/\Phi\mathfrak{g}_0)$. Let $\rho : \mathfrak{g}_0 \rightarrow \mathfrak{gl}(\mathfrak{g}_1/\Phi\mathfrak{g}_0)$ be the obvious representation of the Lie algebra \mathfrak{g}_0 . Let $Q = \mathfrak{g}_1/(\Phi\mathfrak{g}_0 + \mathfrak{h}_1)$. Let $R : E \rightarrow (\mathfrak{g}_0/\mathfrak{h}_0)^* \otimes Q^* \otimes (\mathfrak{g}_1/\Phi\mathfrak{g}_0)$ be the map

$$R(e)(A, B) = qK_1(F(e))(\overline{\Phi A} \wedge \bar{C}),$$

where $qC = B$. This function R is well defined, and called the *morphism curvature* or *Φ -curvature*. The function a restricted to a leaf $E_0 \subset E$ of the morphism deformation satisfies the *equation of first variation*:

$$\nabla a = R(\overline{\Phi\omega_0}, \bar{a}).$$

Remark 34. The morphism curvature is analogous to the Riemann curvature tensor appearing in the Jacobi vector field equation of a geodesic in a pseudo-Riemannian manifold; see example 31 on page 41.

Proof. The first order deformation is $\delta F = qF^*\omega_1$. It vanishes on VE , a corank 1 subbundle of TE . Clearly $VE = (dt = 0)$, so $\delta F = a dt$ for a unique function a , and clearly $a : E \rightarrow \mathfrak{g}_1/\Phi\mathfrak{g}_0$. The H_0 -equivariance of a follows immediately from that of ω_1 .

$$\begin{aligned} d(a dt) &= da \wedge dt \\ &= d\delta F \\ &= dqF^*\omega_1 \\ &= qF^*\left(-\frac{1}{2}[\omega_1, \omega_1] + \frac{1}{2}K_1\bar{\omega}_1 \wedge \bar{\omega}_1\right). \end{aligned}$$

Each vector tangent to E_0 has the form \vec{A} for some unique $A \in \mathfrak{g}_0$.

$$\begin{aligned} \vec{A} \lrcorner (da \wedge dt) &= (\vec{A} \lrcorner da) dt \\ &= q(-\text{Ad}(A)F^*\omega_1 + K_1(\overline{\Phi A}, F^*\bar{\omega}_1)) \\ &= -\rho(A)q\omega_1 + qK_1(\overline{\Phi A}, F^*\bar{\omega}_1) \\ &= -\rho(A)a dt + qK_1(\overline{\Phi A}, F^*\bar{\omega}_1). \end{aligned}$$

If we pick some vector tangent to E_0 , say \vec{B} for some $B \in \mathfrak{g}_0$, then

$$\begin{aligned} 0 &= \vec{B} \lrcorner ((\overline{\Phi A} \lrcorner da) dt) \\ &= \vec{B} \lrcorner (-\rho(A)a dt + qK_1(\overline{\Phi A}, F^*\bar{\omega}_1)) \\ &= qK_1(\overline{\Phi A}, \overline{\Phi B}) \\ &= q\Phi K_0(\bar{A}, \bar{B}) \\ &= 0. \end{aligned}$$

Therefore the expression

$$qK_1(\overline{\Phi A}, F^*\bar{\omega}_1)$$

vanishes on vectors tangent to E_0 , and so is defined on vectors in $TE|_{E_0}/TE_0$, and is expressible in terms of $qF^*\omega_1$, i.e. in terms of $a dt$, as

$$qK_1(\overline{\Phi A}, F^*\bar{\omega}_1) = R(\overline{\Phi A}, \bar{a}) dt.$$

Hence R is well defined, and the equation of first variation follows. \square

Next, we would like a notion of infinitesimal deformation which does not depend on the existence of any actual deformation.

Definition 58. Consider a local model morphism $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are Cartan geometries and that $F : E_0 \rightarrow E_1$ is a Φ -morphism. Let $\rho : \mathfrak{g}_0 \rightarrow \mathfrak{gl}(\mathfrak{g}_1/\Phi\mathfrak{g}_0)$ be the obvious representation of the Lie algebra \mathfrak{g}_0 . Let $Q = \mathfrak{g}_1/(\Phi\mathfrak{g}_0 + \mathfrak{h}_1)$. Let $R : E_0 \rightarrow (\mathfrak{g}_0/\mathfrak{h}_0)^* \otimes Q^* \otimes (\mathfrak{g}_1/\Phi\mathfrak{g}_0)$ be the map

$$R(e)(A, B) = qK_1(F(e))(\overline{\Phi A} \wedge \bar{C}),$$

where $qC = B$. Again we will refer to R as the Φ -curvature, or *morphism curvature*.

An *infinitesimal one-parameter morphism deformation* of F is a section a of $E \times_{H_0} (\mathfrak{g}_1/\Phi\mathfrak{g}_0)$ satisfying the equation of first variation

$$\nabla a = R(\overline{\Phi\omega_0}, \bar{a}).$$

Remark 35. Since the equation of first variation is a total differential equation, each solution is completely determined by its value at any chosen point. In particular, the dimension of the space of global solutions is at most the dimension of $\mathfrak{g}_1/\Phi\mathfrak{g}_0$. Formally we can think of the space of infinitesimal morphism deformations as the tangent space to the space of morphism deformations.

Example 30. Consider a morphism of homogeneous spaces $\Phi : G_0/H_0 \rightarrow G_1/H_1$. The infinitesimal deformations of the model are just the global solutions of $\nabla a = 0$. We can take any element $a_0 \in \mathfrak{g}_1/\Phi\mathfrak{g}_0$, and then let $a(g_0) = \rho(g_0)^{-1} a_0$. It is easy to see that every infinitesimal morphism deformation has this form, so the space of infinitesimal morphism deformations of the model $G_0/H_0 \rightarrow G_1/H_1$ is precisely $\mathfrak{g}_1/\Phi\mathfrak{g}_0$.

Remark 36. To each infinitesimal deformation a of an immersive morphism $F : E_0 \rightarrow E_1$ of Cartan geometries, there is associated a section of the normal bundle F^*TE_1/TE_0 , given by using the Cartan connection ω_1 on E_1 to identify the normal bundle with $E_0 \times_{H_0} (\mathfrak{g}_1/\Phi\mathfrak{g}_0)$.

Lemma 29. Suppose that $E \rightarrow M \rightarrow S$ is a family of Cartan geometries, say with maps $\pi : E \rightarrow M$ and $p : M \rightarrow S$. For each point $s \in S$, let $M_s = p^{-1}s$ and $E_s = \pi^{-1}p^{-1}s$. Suppose that $F : E \rightarrow E_1$ is an immersive morphism deformation. Then at each point $s \in S$, to each tangent vector $\dot{s} \in T_s S$, we can associate a unique infinitesimal deformation $a = a_{\dot{s}} : E_s \rightarrow \mathfrak{g}_1/\Phi\mathfrak{g}_0$ so that, identifying a with a section A of the normal bundle F^*TE_1/TE_s (as in remark 36), and taking any path $e(t) \in E$ for which $(p \circ \pi \circ e)'(0) = \dot{s}$, we have

$$A(e(0)) = F'(e(0))e'(0) + F'(e(0))V_{e(0)}E.$$

We call this $A = A_{\dot{s}}$ the *associated section of the normal bundle*.

Proof. We have to show that we can construct an infinitesimal deformation a with

$$a(e(0)) = F'(e(0))e'(0) \lrcorner q\omega_1.$$

Take any path $s(t)$ in S with $s'(0) = \dot{s}$, and replace S by an interval of this path, and E and M by their respective pullbacks via the map $t \mapsto s(t)$. Then we have to prove that there is some a so that

$$a \, dt = qF^* \omega_1.$$

But clearly since $qF^* \omega_1 = a \, dt$ for a unique infinitesimal deformation a as seen above, this choice of a is precisely the one we must make.

Now consider returning to a general parameter space S . We can pick out a choice of a as above, depending on a choice of path $s(t)$. We need to show that this choice will satisfy

$$a(e(0)) = F'(e(0)) e'(0) \lrcorner q\omega_1$$

for any path $e(t)$ in E for which $(p \circ \phi \circ e)'(0) = \dot{s}$. This certainly works as long as $e(t)$ is a path inside E lying over the curve $s(t)$.

So now we need only show that this infinitesimal deformation a depends only on \dot{s} , not on the particular path $s(t)$. So suppose that we have two paths $s_1(t)$ and $s_2(t)$ in S , with $s'_1(0) = s'_2(0) = \dot{s}$. We then make infinitesimal deformations a_1 and a_2 so that if $e_1(t)$ is a path in the pullback of E over $s_1(t)$, then

$$a_1(e_1(0)) = F'(e_1(0)) e'_1(0) \lrcorner q\omega_1,$$

and similarly for $e_2(t)$. Assume that $e_1(0) = e_2(0) = e_0$ and that $(p \circ \pi \circ e_1)'(0) = (p \circ \pi \circ e_2)'(0) = \dot{s}$. So $e'_1(0) - e'_2(0) \in V_{e_0} E$. So

$$\begin{aligned} a_1(e_1(0)) - a_2(e_2(0)) &= (e'_1(0) - e'_2(0)) \lrcorner qF^* \omega_1(e_0) \\ &= 0. \end{aligned}$$

□

Lemma 30. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is an immersive local model morphism. Suppose that $E_1 \rightarrow M_1$ is a (H_1, \mathfrak{g}_1) -geometry with vanishing Φ -obstruction. Suppose that $E \rightarrow E_1$ is the universal Φ -morphism deformation, and has a parameter space S . The map $\dot{s} \mapsto A_{\dot{s}}$ taking a tangent vector on the parameter space to its associated section of the normal bundle is a linear isomorphism from tangent vectors to TS to infinitesimal morphism deformations.

Proof. Let's start by finding the kernel. If $A_{\dot{s}} = 0$, then taking any path $e(t) \in E$ for which $(p \circ \pi \circ e)'(0) = \dot{s}$, we have

$$0 = A(e(0)) = F'(e(0)) e'(0) + F'(e(0)) V_{e(0)} E.$$

So F' is not an immersion. But $F : E \rightarrow E_1$ must be an immersion by lemma 27 on page 36.

The universal deformation has the same dimension as the model, and its parameter space is therefore the same dimension as $\mathfrak{g}_1/\Phi\mathfrak{g}_0$, and so the linear map $\dot{s} \mapsto A_{\dot{s}}$ is a linear isomorphism. □

Example 31. We continue example 28 on page 37: geodesics of projective connections. We can write any element A in the various Lie algebras in the manner

indicated:

Lie algebra	Also known as	Typical element
\mathfrak{g}_1	$\mathfrak{sl}(n+1, \mathbb{R})$	$\begin{pmatrix} A_0^0 & A_1^0 & A_J^0 \\ A_0^1 & A_1^1 & A_J^1 \\ A_0^I & A_1^I & A_J^I \end{pmatrix}$
\mathfrak{h}_1	stabilizer of a point	$\begin{pmatrix} A_0^0 & A_1^0 & A_J^0 \\ 0 & A_1^1 & A_J^1 \\ 0 & A_1^I & A_J^I \end{pmatrix}$
\mathfrak{g}_0	stabilizer of a line	$\begin{pmatrix} A_0^0 & A_1^0 & A_J^0 \\ A_0^1 & A_1^1 & A_J^1 \\ 0 & 0 & A_J^I \end{pmatrix}$
\mathfrak{h}_0	stabilizer of a pointed line	$\begin{pmatrix} A_0^0 & A_1^0 & A_J^0 \\ 0 & A_1^1 & A_J^1 \\ 0 & 0 & A_J^I \end{pmatrix}$

with indices $I, J = 2, \dots, n$.

The Cartan geometry of a projective connection looks like

$$\omega_1 = \begin{pmatrix} \omega_0^0 & \omega_1^0 & \omega_J^0 \\ \omega_0^1 & \omega_1^1 & \omega_J^1 \\ \omega_0^I & \omega_1^I & \omega_J^I \end{pmatrix}.$$

The quotient $\bar{\omega}_1 = \omega_1 + \mathfrak{h}_1$ looks like

$$\bar{\omega}_1 = \begin{pmatrix} \omega_0^1 \\ \omega_0^I \end{pmatrix}.$$

The curvature of a projective connection looks like

$$K_1 \bar{\omega}_1 \wedge \bar{\omega}_1 = \begin{pmatrix} \nabla \omega_0^0 & \nabla \omega_1^0 & \nabla \omega_J^0 \\ \nabla \omega_0^1 & \nabla \omega_1^1 & \nabla \omega_J^1 \\ \nabla \omega_0^I & \nabla \omega_1^I & \nabla \omega_J^I \end{pmatrix}$$

where

$$\nabla \omega_\nu^\mu = 2K_{\nu 1 I}^\mu \omega_0^1 \wedge \omega_0^I + K_{\nu I J}^\mu \omega_0^I \wedge \omega_0^J$$

for $\mu, \nu = 0, 1, 2, \dots, n$. The first order variation δF of the universal deformation is

$$\begin{aligned} \delta F &= qK_1 \bar{\omega}_1 \wedge \bar{\omega}_1 \\ &= (\nabla \omega_0^I \quad \nabla \omega_1^I) \\ &= (2K_{01 I}^I \omega_0^1 \wedge \omega_0^I + K_{0 I J}^I \omega_0^I \wedge \omega_0^J \quad 2K_{11 I}^I \omega_0^1 \wedge \omega_0^I + K_{1 I J}^I \omega_0^I \wedge \omega_0^J). \end{aligned}$$

The morphism curvature of the universal deformation is

$$R(A_0^1, B_0^I) = (K_{01 J}^I A_0^1 B_0^J \quad K_{11 J}^I A_0^1 B_0^J).$$

An infinitesimal deformation along one of the morphisms (i.e. leaves) of the universal deformation has the form

$$a = (a_0^I \quad a_1^I).$$

The equation of first variation is

$$\nabla (a_0^I \quad a_1^I) = (K_{01 J}^I \omega_0^1 a_0^J \quad K_{11 J}^I \omega_0^1 a_0^J).$$

The representation ρ is

$$\rho(A) \begin{pmatrix} B_0^I & B_1^I \end{pmatrix} = (A_J^I B_0^J - B_0^I A_0^0 - B_1^I A_0^1 \quad A_J^I B_1^J - B_0^I A_1^0 - B_1^I A_1^1).$$

Therefore the covariant derivative ∇a of any infinitesimal deformation a is

$$\nabla \begin{pmatrix} a_0^I & a_1^I \end{pmatrix} = \begin{pmatrix} \omega_J^I a_0^J - a_0^I \omega_0^0 - a_1^I \omega_0^1 & \omega_J^I a_1^J - a_0^I \omega_1^0 - a_1^I \omega_1^1 \end{pmatrix}.$$

Finally, we obtain the equation of first variation of the universal deformation:

$$\begin{aligned} da_0^I &= a_0^I \omega_0^0 + a_1^I \omega_0^1 - \omega_J^I a_0^J + K_{01J}^I \omega_0^1 a_0^J \\ da_1^I &= a_0^I \omega_1^0 + a_1^I \omega_1^1 - \omega_J^I a_1^J + K_{11J}^I \omega_0^1 a_0^J. \end{aligned}$$

The projective connection is torsion-free if and only if $K_{01J}^I = 0$. Therefore a torsion-free projective connection has equation of first variation of its universal deformation:

$$\begin{aligned} da_0^I &= a_0^I \omega_0^0 + a_1^I \omega_0^1 - \omega_J^I a_0^J \\ da_1^I &= a_0^I \omega_1^0 + a_1^I \omega_1^1 - \omega_J^I a_1^J + K_{11J}^I \omega_0^1 a_0^J, \end{aligned}$$

which we can see is naturally a 2nd order system of ordinary differential equations along each geodesic: precisely the well known Jacobi vector field equations.

Example 32. We leave the reader to generalize the previous example to totally geodesic projectively flat subspaces inside a manifold with projective connection. This is really just a slight change of notation from the previous example; see Cartan [8] *Leçons sur la théorie des espaces à connexion projective*, chapter VI.

Example 33. Let's continue with example 29 on page 37: $G_1/H_1 = \mathcal{O}(n)$, $G_1 = (\mathrm{GL}(2, \mathbb{R})/C_n) \rtimes \mathrm{Sym}^n(\mathbb{R}^2)^*$. It is convenient to write elements $(g, p) \in G_1$ formally as matrices

$$\begin{pmatrix} g & p \\ 0 & 1 \end{pmatrix}$$

to make the semidirect product structure more apparent. In this setting, we write gp to mean $p \circ g^{-1}$, as a homogeneous polynomial. Clearly

$$\begin{pmatrix} g & p \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} g^{-1} & -g^{-1}p \\ 0 & 1 \end{pmatrix}.$$

We will write any homogeneous polynomial $p \in \mathrm{Sym}^n(\mathbb{C}^2)^*$ as

$$p(x, y) = \sum_{i+j=n} p_{ij} x^i y^j.$$

Suppose that

$$\begin{array}{ccc} H_1 & \longrightarrow & E_1 \\ & & \downarrow \pi \\ & & M_1, \end{array}$$

is an $\mathcal{O}(n)$ -geometry on a surface M_1 , with Cartan connection $\omega_1 \in \Omega^1(E_1) \otimes \mathfrak{g}$. We can write

$$\omega = \begin{pmatrix} \gamma & \varpi \\ 0 & 0 \end{pmatrix},$$

with $\gamma \in \Omega^1(E) \otimes \mathfrak{gl}(2, \mathbb{C})$ and $\varpi \in \Omega^1(E) \otimes \mathrm{Sym}^n(\mathbb{C}^2)^*$, say

$$\gamma = \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \\ \gamma_1^2 & \gamma_2^2 \end{pmatrix}$$

and

$$\varpi = (\varpi_{ij}),$$

where we write each homogeneous polynomial $p(x, y)$ as

$$p(x, y) = \sum_{i+j=n} p_{ij} x^i y^j.$$

Under H_1 -action on E_1 , ω_1 varies in the adjoint representation, $r_h^* \omega = \text{Ad}_h^{-1} \omega_1$. Therefore, if we write

$$h = \begin{pmatrix} g & p \\ 0 & 1 \end{pmatrix}$$

then

$$r_h^* \begin{pmatrix} \gamma & \varpi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} g^{-1} \gamma g & g^{-1} \gamma p + g^{-1} \varpi \\ 0 & 0 \end{pmatrix}.$$

The action on ϖ , expanded out, when we think of ϖ as a 1-form valued in symmetric n -linear forms, is

$$\begin{aligned} (g^{-1} \gamma p + g^{-1} \varpi)(v_1, v_2, \dots, v_n) &= - \sum_k p(v_1, v_2, \dots, v_{k-1}, g^{-1} \gamma v_k, v_{k+1}, \dots, v_n) \\ &\quad + \varpi(gv_1, gv_2, \dots, gv_n). \end{aligned}$$

The subalgebra $\mathfrak{h}_1 \subset \mathfrak{g}_1$ consists in the elements for which $\gamma_1^2 = 0$ and $\varpi_{n0} = 0$. Therefore we can treat the pair

$$(\gamma_1^2, \varpi_{n0})$$

as being a 1-form valued in $\mathfrak{g}_1/\mathfrak{h}_1$. In particular, we can write our structure equations as

$$d \begin{pmatrix} \gamma & \varpi \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \gamma & \varpi \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \gamma & \varpi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} K & L \\ 0 & 0 \end{pmatrix} \gamma_1^2 \wedge \varpi_{n0},$$

where $K : E_1 \rightarrow \Lambda^2(\mathfrak{g}_1/\mathfrak{h}_1)^* \otimes \mathfrak{gl}(2, \mathbb{C})$ and $W : E_1 \rightarrow \Lambda^2(\mathfrak{g}_1/\mathfrak{h}_1)^* \otimes \text{Sym}^n(\mathbb{C}^2)^*$. The invariants $W = (W_{ij})$ are called the *Wilczynski invariants*. Expanding this out gives

$$\begin{aligned} d\gamma + \gamma \wedge \gamma &= K \gamma_1^2 \wedge \varpi_{n0}, \\ d\varpi + \gamma \wedge \varpi &= W \gamma_1^2 \wedge \varpi_{n0}. \end{aligned}$$

It is convenient to write $\varpi = (\varpi_{ij})$ corresponding to the expression for polynomials $p(x, y) = \sum_{i+j=n} p_{ij} x^i y^j$. In terms of such an expression,

$$d\varpi_{ij} = \gamma_1^1 \wedge \varpi_{ij} + \gamma_1^2 \wedge \varpi_{i-1, j+1} + L_{ij} \gamma_1^2 \wedge \varpi_{n0}.$$

Consider the morphism of homogeneous spaces $\Phi : G_0/H_0 \rightarrow G_1/H_1$, where $G_0 = \text{GL}(2, \mathbb{R})/C_n$ and $H_0 = G_0 \cap G_1$. The Φ -torsion vanishes on any G_1/H_1 -geometry, because G_0/H_0 is 1-dimensional. The Φ -morphism curvature is precisely $R = L$: the morphism curvature is the collection of Wilczynski invariants. In particular, the scalar ordinary differential equations which satisfy $L = 0$ are of obvious interest, since their first order deformations are the same as those of the model equation $d^{n+1}y/dx^{n+1} = 0$. The scalar ordinary differential equations with $L = 0$ (vanishing Wilczynski invariants) are well known and geometrically characterized by Dunajski and Tod [13] and Doubrov [11].

18. THE CANONICAL LINEAR SYSTEM OF A MORPHISM

Definition 59. Suppose that (H, \mathfrak{g}) is a local model, V a \mathfrak{g} -module, V' an H -module, $\lambda : V \rightarrow V'$ a morphism of H -modules, and $E \rightarrow M$ a (H, \mathfrak{g}) -geometry. The *locus map* f_λ is the map associating to each point $m \in M$ the set of parallel sections of $E \times_H V$ whose image under $E \times_H V \rightarrow E \times_H V'$ vanishes at m , so $f_\lambda : M \rightarrow \bigcup_{k=0}^{\dim V} \text{Gr}(k, Z)$ where Z is the space of parallel sections of $E \times V \rightarrow M$.

Example 34. Suppose that G/H is a homogeneous space. Let K be the kernel of $\lambda : V \rightarrow V'$. The model locus map is the map taking $gH \in G/H \mapsto gK \in \text{Gr}(\dim K, V)$. The model locus map is an immersion just when the Lie algebra of the stabilizer of K in G is contained in the Lie algebra of H .

Definition 60. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are Cartan geometries and $F : E_0 \rightarrow E_1$ is a Φ -morphism. Then the locus map Λ_F of the morphism F is the locus map of the H_0 -module morphism $\lambda : \mathfrak{g}_1/\Phi\mathfrak{g}_0 \rightarrow \mathfrak{g}_1/(\mathfrak{h}_1 + \Phi\mathfrak{g}_0)$.

The locus map of a morphism associates to each point m_0 of the source M_0 the set of all infinitesimal deformations whose underlying normal bundle section vanishes at m_0 .

Example 35. Suppose that $\Phi : G_0/H_0 \rightarrow G_1/H_1$ is a morphism of homogeneous spaces. Consider the locus map of Φ as a Φ -morphism of Cartan geometries. This locus map is a map $\Lambda_\Phi : G_0/H_0 \rightarrow \text{Gr}(\dim K, \mathfrak{g}_1/\Phi\mathfrak{g}_0)$ where $K = \mathfrak{h}_1 + \Phi\mathfrak{g}_0 \subset \mathfrak{g}_1$. It is defined as

$$\Lambda_\Phi(g_0H_0) = g_0K \subset \mathfrak{g}_1.$$

Let \mathfrak{k} be the set of all $A \in \mathfrak{g}_1$ for which $\text{Ad}(A)K \subset K$. The locus map is an immersion just when $\mathfrak{k} \subset \mathfrak{h}_1$.

Definition 61. The *expected dimension* of the space of infinitesimal deformations of a morphism of Cartan geometries is the dimension of the space of infinitesimal deformations of the model.

We can apply all of the ideas we developed so far to prove completeness results.

Proposition 10. Suppose that $\Phi : G_0/H_0 \rightarrow G_1/H_1$ is a morphism of homogeneous spaces. Suppose that the model locus map is an immersion. Suppose that $E_0 \rightarrow M_0$ is a flat G_0/H_0 -geometry on a compact manifold M_0 . Suppose that $F : E_0 \rightarrow E_1$ is a Φ -morphism, whose infinitesimal deformations have expected dimension, and whose deformation curvature vanishes. Then the locus map of F is a covering map to its image and the geometry $E_0 \rightarrow M_0$ is complete.

Proof. Because M_0 is flat, it is locally isomorphic to the model. Because the deformation curvature vanishes, each local isomorphism identifies the infinitesimal deformations and their vanishing loci, and therefore near each point, the locus map is identified locally. Because M_0 is compact, the image $\Lambda_F M_0$ of the locus map of M_0 is identical to that of G_0/H_0 , a G_0 -orbit inside the appropriate Grassmannian; let's call it G_0/H'_0 . Moreover, the locus map is an immersion, because it is an immersion of the model. In particular, $H'_0 = H_0 \rtimes \Gamma$, for some discrete group Γ . By homogeneity, the image G_0/H'_0 of either locus map is an embedded submanifold of the Grassmannian. By compactness of M_0 , the image G_0/H'_0 is a compact manifold, and so the locus map is a finite covering map $\Lambda_F : M_0 \rightarrow G_0/H'_0$. By

homogeneity, the locus map of the model is a covering map $\Lambda_\Phi : G_0/H_0 \rightarrow G_0/H'_0$. Let \tilde{M}_0 be the pullback:

$$\begin{array}{ccc} \tilde{M}_0 & \longrightarrow & G_0/H_0 \\ \downarrow & & \downarrow \\ M_0 & \longrightarrow & G_0/H'_0. \end{array}$$

Clearly \tilde{M}_0 bears the pullback Cartan geometry from M_0 , and the maps from \tilde{M}_0 to M_0 and to G_0/H_0 are local isomorphisms of Cartan geometries. Indeed theorem 9 on page 24 already furnishes us this same manifold \tilde{M}_0 .

Take any element $A \in \mathfrak{g}_0$. Look at the vector field X_A on G_0/H'_0 which is the infinitesimal left action of A . This vector field lifts via the locus map, since the locus map is a covering map, to a vector field on M_0 . It also lifts in the obvious way to a vector field on G_0/H_0 . (Keep in mind that this is *not* the constant vector field that comes from the model Cartan connection. Instead it is an infinitesimal symmetry of the flat Cartan geometry.) This vector field on M_0 is locally identified with the corresponding field on G_0/H_0 , via any local isomorphism given by locally inverting the locus maps. The flows intertwine in the obvious way, so that the completeness of flows on the homogeneous spaces G_0/H_0 and G_0/H'_0 ensures the completeness of X_A on all four manifolds:

$$\begin{array}{ccc} \tilde{M}_0 & \longrightarrow & G_0/H_0 \\ \downarrow & & \downarrow \\ M_0 & \longrightarrow & G_0/H'_0. \end{array}$$

These vector fields generate a locally transitive action of \mathfrak{g}_0 on all four manifolds. By a theorem of Ehresmann [15] or McKay [30], each of these maps is a fiber bundle mapping, so a covering map. Completeness of Cartan geometries is preserved and reflected by covering maps so the Cartan geometry on M_0 is complete. \square

Corollary 12. Suppose that $\Phi : G_0/H_0 \rightarrow G_1/H_1$ is an immersive morphism of homogeneous spaces. Suppose that the locus map of G_0/H_0 is an immersion (see example 35 on the preceding page). Suppose that $E_1 \rightarrow M_1$ is a G_1/H_1 -geometry with vanishing Φ -obstruction, vanishing deformation curvature, and is Φ -tame. Then the vector fields \vec{A} on E , for $A \in \Phi\mathfrak{g}_0$, are complete.

Example 36.

Theorem 14 (McKay [29]). Every normal projective connection on a surface, all of whose geodesics are embedded closed curves, is tame and complete.

Proof. A projective connection on a surface is a Cartan geometry modelled on $G/H = \mathbb{RP}^2$ with $G = \mathbb{P}\mathrm{SL}(3, \mathbb{R})$. It is normal just when certain curvature terms vanish; see McKay [29]. The morphism curvature vanishes, as we can easily check. If it is tame then it is complete by corollary 12. By a theorem of LeBrun and Mason [27], a projective connection is tame just precisely when all of the geodesics are closed and embedded. \square

Remark 37. LeBrun and Mason [27] prove that the only normal projective connection on \mathbb{RP}^2 all of whose geodesics are closed and embedded curves is the usual flat connection, i.e. on \mathbb{RP}^2 tame implies flat and isomorphic to the model.

Example 37. Let's use the previous example to see that tameness is strictly stronger than completeness.

Theorem 15 (McKay [29]). On any compact manifold, any pseudo-Riemannian metric with positive Ricci curvature induces a complete normal projective connection.

Corollary 13. A smooth compact surface of positive Gauss curvature has complete normal projective connection, but is not tame unless it is a Zoll surface.

Since Zoll surfaces are quite rare, we can already see that tameness is much stronger than completeness.

Example 38.

Definition 62. A G/H -structure is a flat G/H -geometry.

Theorem 16 (Sharpe [36]). A G/H -structure on a manifold M is complete if and only if $M = \Gamma \backslash \tilde{G}/H$, where \tilde{G} is a Lie group containing H and \tilde{G}/H has the same local model as G/H and $\Gamma \subset \tilde{G}$ is a discrete subgroup acting freely and properly on \tilde{G}/H .

Remark 38. Hence classification of such structures reduces to algebra.

Corollary 14. If a surface M has a flat projective connection, all of whose geodesics are embedded closed curves, then the surface is the sphere or real projective plane with the usual flat projective connection.

Proof. By theorem 14 on the facing page, the projective connection is complete. Flat and complete implies isomorphic to the model up to covering, by theorem 16. \square

Example 39. A totally geodesic surface is a morphism of Cartan geometries to any manifold with projective connection, in the obvious way: in terms of Cartan's structure equations for projective connections (see Kobayashi and Nagano [24]).

Theorem 17. If a normal projective connection on a manifold has all geodesics closed and, for some integer $k > 1$, contains a compact totally geodesic submanifold of dimension k through each point tangent to each k -plane, then it is a lens space $\Gamma \backslash S^n$, where Γ is a finite group of orthogonal transformations of \mathbb{R}^{n+1} , acting freely on the unit sphere S^n .

Proof. It is easy to see that the torsion for such morphisms is just the curvature of the projective connection. Therefore the existence of such a large family of totally geodesic surfaces immediately implies that the curvature vanishes.

The induced projective connections on the totally geodesic surfaces are normal, because the ambient projective connection is normal. The compactness of each surface ensures that their induced projective connections are all complete. But then applying corollary 12 on the facing page, we find immediately that the ambient projective connection is complete.

Completeness and flatness implies by theorem 16 that M is the model up to covering. Since the universal covering of the model is the sphere, M is covered by the sphere, say $M = \Gamma \backslash S^n$, where Γ is a discrete group acting by projective automorphisms on S^n , freely and properly. Since S^n is compact, Γ is finite, and therefore Γ is, up to projective conjugacy, a subgroup of the maximal compact subgroup of the projective automorphism group of S^n , i.e. Γ is a group of orthogonal transformations. \square

A projective Blaschke conjecture:

Conjecture 2. If a normal projective connection has all its geodesics closed embedded curves, then it is the standard normal projective connection of a compact rank one symmetric space.

Example 40. Continuing with example 33 on page 43, take a $\mathcal{O}(n)$ -geometry with vanishing Wilczynski invariants. In that example, we saw that the Φ -torsion for the obvious morphism Φ vanishes and we identified the morphism curvature with the Wilczynski invariants. If the $\mathcal{O}(n)$ -geometry is tame then it is complete by corollary 12 on page 46. It isn't clear how to classify the flat complete or the flat tame $\mathcal{O}(n)$ -geometries.

Conjecture 3. An $\mathcal{O}(n)$ -geometry is tame if and only if it is isomorphic to a covering space of the model.

Example 41. For certain types of model there is a purely algebraic description of the flat tame geometries with that model.

Proposition 11. Suppose that $\Phi : G_0/H_0 \rightarrow G_1/H_1$ is an immersive morphism of homogeneous spaces. Suppose that the locus map of G_0/H_0 is an immersion (see example 35 on page 45). Suppose that $E_1 \rightarrow M_1$ is a Φ -amenable G_1/H_1 -structure. Then the vector fields \vec{A} on E , for $A \in \Phi\mathfrak{g}_0$, are complete. If moreover $\mathfrak{h}_1 + \Phi\mathfrak{g}_0$ generates \mathfrak{g}_1 as a Lie algebra, then M_1 is complete. In particular, $M_1 = \Gamma \backslash \tilde{G}_1/H_1$ as in theorem 16 on the previous page.

Proof. Corollary 12 on page 46 applies because all of the curvature of M_1 vanishes. If moreover $\mathfrak{h}_1 + \Phi\mathfrak{g}_0$ generates \mathfrak{g}_1 as a Lie algebra, then M_1 is flat and complete, so the rest follows from theorem 16 on the previous page. \square

Example 42. The obvious inclusion $\mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{SL}(n+1, \mathbb{R})$ as linear transformations of a 2-plane fixing a complementary $(n-1)$ -plane yields a morphism $\Phi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^n$. We will say that a projective connection is *tame* if it is Φ -tame.

Theorem 18. A manifold with flat complete projective connection is a lens space $\Gamma \backslash S^n$ where S^n is the sphere with its usual flat projective connection (the Levi-Civita connection of the usual round Riemannian metric), and $\Gamma \subset \mathrm{O}(n+1)$ is a finite subgroup of the orthogonal group $\mathrm{O}(n+1)$ acting freely on S^n .

A manifold with flat tame projective connection is S^n or \mathbb{RP}^n with its usual flat projective connection.

Proof. By theorem 16 on the preceding page, if a manifold has a flat complete projective connection, it must be a quotient of the universal covering space of the model, i.e. of the sphere S^n with its usual flat projective connection, quotiented by a discrete group Γ of automorphisms of its projective connection. Since the sphere is compact, Γ must be a finite group. The automorphisms of the projective connection are the invertible linear maps (modulo scaling by positive constants). We can assume that Γ lies in the maximal compact subgroup of the automorphism group, the orthogonal group.

To see when such a quotient is tame, we need to check that the action of Γ on the Grassmannian of oriented 2-planes in \mathbb{R}^{n+1} has no fixed points, or has only fixed points. This happens just for $\Gamma = \{I\}$ or $\Gamma = \{\pm I\}$. \square

Example 43. A flat conformal geometry has lots of curves which are locally identified with 2-spheres in the model. (Some of these have to be thought of as 2-spheres of zero radius, in order to present them as morphisms.) Clearly tameness of these objects implies completeness of the conformal geometry, and so each path component of the manifold is a quotient of the model, with induced conformal geometry, i.e. $\Gamma \backslash S^n$, where Γ is a finite group of rotations. We leave the reader to work out the relevant Lie algebras, homogeneous spaces and morphism.

Example 44. A k -plane field is a rank k subbundle of the tangent bundle of a manifold. If P is a sheaf of vector fields on a manifold M , let $P^{(1)}$ be the sheaf of locally defined vector fields spanned by local sections of P together with Lie brackets $[Y, Z]$ of local sections of P . Define $P^{(2)}, P^{(3)}, \dots$ by induction:

$$P^{(j+1)} = \left(P^{(j)} \right)^{(1)}.$$

Let's say that a 2-plane field P on a 5-manifold is *nondegenerate* if $P^{(1)}$ is a 3-plane field and $P^{(2)} = TM$.

Let G_2 be the adjoint form of the split form of the exceptional simple group of dimension 14. A nondegenerate 2-plane field on a 5-manifold M gives rise to a Cartan geometry $E \rightarrow M$ modelled on G_2/P_2 for a certain parabolic subgroup $P_2 \subset G_2$; see [6, 20, 38]. (The subgroup P_2 is unique up to conjugacy.) We say the plane field is *flat* if this Cartan geometry is flat. Cartan proved that a nondegenerate 2-plane field is flat if and only if it is locally isomorphic to the model 2-plane field on the model G_2/P_2 . (This model 2-plane field is the unique G_2 -invariant 2-plane field on G_2/P_2 . The group of diffeomorphisms of G_2/P_2 preserving that 2-plane field is a Lie group and has the same Lie algebra as G_2 , although it is strictly larger than G_2 . It has at least 4 components.)

Let $B \subset G_2$ be a Borel subgroup contained in P_2 , and let $P_1 \subset G_2$ be the parabolic subgroup of G_2 containing B and not conjugate to P_2 (see [19] for proof that P_1 exists and is unique and for an explicit description of P_1 .) Then E/B has a foliation by curves: the leaves of the exterior differential system $\omega + \mathfrak{p}_1 = 0$. These leaves are curves, known as *characteristics* [6] or *singular extremals* [4]. Let W be the space of characteristics. As far as the author knows W might not be Hausdorff. We will say that the 2-plane field is *tame* if W is a smooth manifold and $E/B \rightarrow W$ is a smooth circle bundle (not necessarily a principal circle bundle). Clearly the model is tame. Clearly tameness is Φ -tameness for the obvious choice of $\Phi : P_1/B \rightarrow G_2/P_2$. In particular, by corollary 12 on page 46, tameness implies completeness of the Cartan geometry.

For example, take any two surfaces S and S' with Riemannian metrics. The bundles OS and OS' of orthonormal bases for the tangent spaces of S and S' are each equipped with the usual 1-forms (see [3] p. 89 for example), say $\omega_1, \omega_2, \omega_{12}$ and $\omega'_1, \omega'_2, \omega'_{12}$ satisfying the structure equations of Riemannian geometry:

$$\begin{aligned} d \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} &= - \begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad d\omega_{12} = K\omega_1 \wedge \omega_2, \\ d \begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} &= - \begin{pmatrix} 0 & \omega'_{12} \\ -\omega'_{12} & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix}, \quad d\omega'_{12} = K'\omega'_1 \wedge \omega'_2. \end{aligned}$$

The 3-plane field $\mathcal{I} = (\omega_1 = \omega'_1, \omega_2 = \omega'_2, \omega_{12} = \omega'_{12})$ on $OS \times OS'$ is semibasic for the circle action of rotating the tangent spaces of S and S' simultaneously at the

same rate. Moreover the paths drawn out in $OS \times OS'$ as orbits of this circle action are integral curves of \mathcal{I} . Therefore \mathcal{I} descends to a 2-plane field (also called \mathcal{I}) on the space of orbits. This space of orbits is a 5-manifold, which we call M , and consists precisely in the linear isometries between tangent spaces of S and S' . We can think of the paths of the resulting 2-plane field as “rolling of S along a path on S' ,” or vice versa. Clearly the isometries of S and those of S' act on M preserving the 2-plane field.

It is well known (see [4]) that \mathcal{I} is a nondegenerate just above those points where S and S' have distinct Gauss curvature. There is a natural family of \mathcal{I} -tangent curves: pick an oriented geodesic on S , and a point p on that geodesic, an oriented geodesic on S' and a point p' on that geodesic, and a linear isometry $T_p S \rightarrow T_{p'} S'$ identifying the unit tangent vectors of the geodesics, and extend that identification by parallel transport along both geodesics. It is well known that the Φ -morphisms, i.e. the characteristics, are precisely these \mathcal{I} -tangent curves in M .

Conjecture 4. Suppose that S and S' are surfaces with different Gauss curvatures at any pair of points, and M is the space of linear isometries of tangent planes of S and S' . Then the G_2/P_2 -geometry of the nondegenerate 2-plane field on the space is complete just when S and S' have complete Riemannian metrics.

Conjecture 5. Suppose that S and S' are surfaces with different Gauss curvatures at any pair of points, and M is the space of linear isometries of tangent planes of S and S' . Then the G_2/P_2 -geometry of the nondegenerate 2-plane field on M is tame just when both surfaces are Zoll surfaces (i.e. all of their geodesics are closed and embedded curves) and the length L of all closed geodesics on S is a rational multiple of the length L' of all closed geodesics on S' .

Consider a special case: let S be a sphere of radius 1 and S' be a sphere of radius 3, and let M be the space of linear isometries between tangent spaces of S and S' . It is well known [1, 35] that the model G_2/P_2 is a single component of the space M consisting precisely in the linear isometries which preserve orientation. Clearly the “phase space” manifold M has two components, so consists of two copies of G_2/P_2 . Isometries of the sphere of radius 1 and of the sphere of radius 3 act on M preserving the 2-plane field, so their orientation preserving components sit in G_2 , giving the maximal compact subgroup $\mathrm{SO}(3) \times \mathrm{SO}(3) \subset G_2$. The automorphism group of the 2-plane field contains $\mathrm{O}(3) \times \mathrm{O}(3)$, and has Lie algebra \mathfrak{g}_2 by Cartan’s work [6]. The maximal compact subgroup therefore has Lie algebra $\mathfrak{so}(3) \times \mathfrak{so}(3)$, so the automorphism group of the 2-plane field has at least 4 components. It does not appear to be known in the literature what the automorphism group of Cartan’s example is. It is known [1, 35] that the simply connected double cover of G_2/P_2 is $S^2 \times S^3$.

Lemma 31. The characteristics of any flat complete nondegenerate 2-plane field are closed and embedded.

Proof. By theorem 16 on page 47 every flat complete G_2/P_2 -geometry on a connected manifold has the form $M = \Gamma \backslash \tilde{G}_2/P_2$, where Γ is a discrete subgroup of the covering group \tilde{G}_2 of G_2 which acts on the universal covering space of G_2/P_2 . Clearly \tilde{G}_2/P_2 is simply connected and compact. It is enough to check therefore that the characteristics of \tilde{G}_2/P_2 are closed, which is easy to check. They are embedded because they are homogeneous. \square

Theorem 19. There are infinitely many nonhomeomorphic compact 5-manifolds admitting flat tame nondegenerate 2-plane fields, and there are infinitely many nonhomeomorphic compact 5-manifolds admitting flat complete but *not* tame nondegenerate 2-plane fields.

Proof. By theorem 16 on page 47 every flat complete G_2/P_2 -geometry on a connected manifold has the form $M = \Gamma \backslash \tilde{G}_2/P_2$, where Γ is a discrete subgroup of the covering group \tilde{G}_2 of G_2 which acts on the universal covering space of G_2/P_2 . Clearly \tilde{G}_2/P_2 is simply connected and compact. Therefore Γ must be finite. Without loss of generality, Γ is a finite subgroup of the maximal compact subgroup of \tilde{G}_2 , i.e. of $(\mathrm{SU}(2) \times \mathrm{SU}(2)) / \langle (-1, -1) \rangle$.

The finite subgroups of $\mathrm{SU}(2)$ are well known: essentially they are the preimages via $\mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$ of the finite subgroups of $\mathrm{SO}(3)$. To be precise, the finite subgroups of $\mathrm{SU}(2)$, up to conjugacy, are:

- (1) The cyclic group generated by

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix}, \varepsilon = e^{2\pi i/n}$$

- (2) the binary dihedral group generated by

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \varepsilon = e^{2\pi i/n}$$

- (3) the binary tetrahedral group generated by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix}, \varepsilon = e^{2\pi i/8}$$

- (4) the binary octahedral group generated by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix}, \varepsilon = e^{2\pi i/8}$$

- (5) the binary icosahedral group generated by

$$-\begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon + \varepsilon^4 & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix}, \varepsilon = e^{2\pi i/5}.$$

Clearly $G_2/P_2 = (\mathrm{SO}(3) \times \mathrm{SO}(3)) / S_2$ for the diagonal circle subgroup $S_2 \subset \mathrm{SO}(3) \times \mathrm{SO}(3)$, consisting in simultaneous rotations of the two spheres about the vertical axis, in terms of Cartan's picture of spheres of radius 1 and 3, pictured sitting tangent to one another, south pole to north pole. In terms of quaternions, this is the subgroup

$$(e^{\theta k}, e^{\theta k})$$

for $0 \leq \theta < 2\pi$.

Clearly $G_2/P_1 = (\mathrm{SO}(3) \times \mathrm{SO}(3)) / S_1$ for the circle subgroup $S_1 \subset \mathrm{SO}(3) \times \mathrm{SO}(3)$, consisting in rotating one sphere around a horizontal axis, and the other sphere in the opposite direction around the same axis at a third the speed, rolling them on each other so that they travel in opposite directions. In terms of quaternions, this is the subgroup

$$(e^{3\theta j}, e^{-\theta j})$$

for $0 \leq \theta < 2\pi$.

The problem is to figure out which finite subgroups of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ will turn out to give a Haussdorff quotient of one or the other of these 5-manifolds \tilde{G}_2/P_1

and \tilde{G}_2/P_2 , i.e. act freely. Those which act freely on both 5-manifolds are precisely the flat tame 2-plane fields.

So far, the candidate groups Γ are precisely the finite subgroups of $\Gamma_1 \times \Gamma_2$, where $\Gamma_1, \Gamma_2 \subset \mathrm{SO}(3)$ are finite subgroups. We have to check which such groups Γ act freely on G_2/P_1 and which act freely on G_2/P_2 . Let's try G_2/P_2 first. In other words, we need to ensure that if an element

$$\gamma = (\gamma_1, \gamma_2) \in \Gamma$$

fixes an element $(g_1, g_2) S_2 \in (\mathrm{SU}(2) \times \mathrm{SU}(2))/S_2$, then it fixes all such elements. Otherwise the quotient $\Gamma \backslash (\mathrm{SU}(2) \times \mathrm{SU}(2))/S_2$ will not be Hausdorff. Fixed points will be those satisfying

$$\begin{aligned}\gamma_1 g_1 &= g_1 e^{k\theta} \\ \gamma_2 g_2 &= g_2 e^{k\theta},\end{aligned}$$

so that in particular, $g_1^{-1} \gamma_1 g_1 = g_2^{-1} \gamma_2 g_2 = e^{k\theta}$, and so γ_1 and γ_2 are conjugate in $\mathrm{SU}(2)$. Working only up to conjugacy, since every element of $\mathrm{SU}(2)$ is conjugate to something of the form $e^{k\theta}$, an element $(\gamma_1, \gamma_2) \in \Gamma$ has fixed points in G_2/P_2 just when γ_1 and γ_2 are conjugate.

Elements of $\mathrm{SU}(2)$ are conjugate just when they have the same eigenvalues as 2×2 matrices. All elements of $\mathrm{SU}(2)$ are diagonalizable. The eigenvalues of elements of a finite subgroup of $\mathrm{SU}(2)$ can be any root of unity. So finally, Γ acts freely on G_2/P_2 just when, for every element $(\gamma_1, \gamma_2) \in \Gamma$, either γ_1 and γ_2 have distinct eigenvalues, or else $\gamma = (-1, -1)$ or $\gamma = (1, 1)$.

A similar analysis says that Γ acts freely on G_2/P_1 just when, for every element $(\gamma_1, \gamma_2) \in \Gamma$, either γ_1^{-1} and γ_2^3 have distinct eigenvalues, or else $\gamma = (-1, -1)$ or $\gamma = (1, 1)$.

It seems likely that one could analyse all of the possible finite subgroups Γ to determine which act freely on G_2/P_1 and which act freely on G_2/P_2 . We will leave this to the reader and simply give examples of cyclic groups Γ for which these actions are either free or not free. Suppose that Γ is generated by an element $\gamma = (\gamma_1, \gamma_2) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ where γ_1 and γ_2 are of finite order, say with eigenvalues $e^{2\pi i p_1/q_1}$ and $e^{2\pi i p_2/q_2}$ respectively, where p_1, q_1, p_2 and q_2 are integers. Without loss of generality, $0 \leq p_1 < q_1$ and $0 \leq p_2 < q_2$.

In order to ensure that Γ acts freely on G_2/P_2 , we need to ensure precisely that, for an integer k , if

$$k \left(\frac{p_1}{q_1} - \frac{p_2}{q_2} \right)$$

is an integer (i.e. γ_1^k is conjugate to γ_2^k), then

$$k \frac{p_1}{q_1} \text{ and } k \frac{p_2}{q_2}$$

are equal half integers (i.e. γ_1^k and γ_2^k are both ± 1).

There are three possibilities here:

- (1) $p_1 = p_2 = 0$, i.e. $\gamma_1 = \gamma_2 = 1$ or
- (2) $\frac{p_1}{q_1} = \frac{p_2}{q_2} = \frac{1}{2}$ i.e. $\gamma_1 = \gamma_2 = -1$ or
- (3) if we write $\frac{p_1}{q_1} - \frac{p_2}{q_2}$ as $\frac{r}{s}$ in lowest terms, then if s divides kr , we need q_1 to divide $2kp_1$ and q_2 to divide $2kp_2$.

The second possibility is clearly one that leads precisely to Cartan's model. Let's consider the second possibility more carefully.

Since p_1 and q_1 are relatively prime, to have q_1 divide $2kp_1$ is precisely to have q_1 divide $2k$. Therefore Γ acts freely on G_2/P_2 just when for any integer k , if s divides k , then both q_1 and q_2 divide $2k$, i.e. we need q_1 and q_2 to both divide $2s$. This quantity s is just the denominator of $\frac{p_1}{q_1} - \frac{p_2}{q_2}$ in lowest terms, i.e.

$$s = \frac{q_1 q_1}{\gcd(q_1 q_2, p_1 q_2 - q_1 p_2)}.$$

So Γ acts freely precisely when q_1 and q_2 both divide the integer

$$\frac{2q_1 q_1}{\gcd(q_1 q_2, p_1 q_2 - q_1 p_2)}.$$

This occurs just when the least common multiple $\ell = \text{lcm}(q_1, q_2)$ divides

$$\frac{2q_1 q_1}{\gcd(q_1 q_2, p_1 q_2 - q_1 p_2)},$$

i.e. when

$$\frac{q_1 q_2}{\gcd(q_1, q_2)} \text{ divides } \frac{2q_1 q_1}{\gcd(q_1 q_2, p_1 q_2 - q_1 p_2)},$$

i.e. when

$$\gcd(q_1 q_2, p_1 q_2 - q_1 p_2) \text{ divides } 2 \gcd(q_1, q_2).$$

So finally we find that Γ acts freely on G_2/P_2 just when

$$\gcd(q_1 q_2, p_1 q_2 - q_1 p_2) \text{ divides } 2 \gcd(q_1, q_2).$$

and acts freely on G_2/P_1 just when

$$\gcd(q_1 q_2, p_1 q_2 + 3 q_1 p_2) \text{ divides } 2 \gcd(q_1, q_2).$$

For example, if we take any prime number q and let $p_1/q_1 = 2/q$ and $p_2/q_2 = 1/q$, then we will find that Γ acts freely on both G_2/P_1 and G_2/P_2 . Hence there are infinitely many nonhomeomorphic compact 5-manifolds admitting flat tame nondegenerate 2-plane fields.

On the other hand, if we take any prime number $q > 2$ and pick any number $1 \leq p_2 < q$ and let $p_1/q_1 = 1 - 3p_2/q$, then we find that Γ acts freely on G_2/P_2 but does *not* act freely on G_2/P_1 . Hence there are infinitely many nonhomeomorphic compact 5-manifolds admitting flat complete nontame nondegenerate 2-plane fields.

We leave to the reader the problem of classifying all of the possible finite subgroups $\Gamma \subset \text{SU}(2) \times \text{SU}(2)$ which act freely on G_2/P_1 or on G_2/P_2 . \square

Remark 39. It is not known which compact 5-manifolds admit a nondegenerate 2-plane field. It is not even known which compact 5-manifolds admit a *flat* nondegenerate 2-plane field. It is not known whether every nondegenerate 2-plane field can be deformed into every other nondegenerate 2-plane field through a connected family of nondegenerate 2-plane fields.

Remark 40. It is not known how to identify whether a nondegenerate 2-plane field on a 5-manifold is constructed by rolling a surface on another surface, or how invariants of surface geometry appear in the curvature of the G_2/P_2 -geometry.

Remark 41. The computation of the morphism curvature for 2-plane fields has never been done. It might provide some interpretation of the curvature of the G_2/P_2 -geometry of nondegenerate 2-plane fields. I conjecture that the morphism curvature of a nondegenerate 2-plane field vanishes just when the curvature vanishes, i.e. just when the 2-plane field is locally isomorphic to the 2-plane field on G_2/P_2 . Roughly speaking, the curvature of a nondegenerate 2-plane field is conjecturally the infinitesimal deformation equation for characteristics.

Remark 42. It is not known if it is possible to find pairs (M_1, M_2) and (N_1, N_2) of surfaces with Riemannian (or Lorentzian) metrics so that the associated 2-plane field for rolling M_1 on M_2 is (locally) isomorphic to that for rolling N_1 on N_2 , but so that M_1 is not locally isometric to either N_1 or N_2 up to constant scaling.

19. PROOF OF THE ROLLING THEOREMS

Definition 63. Suppose that (H, \mathfrak{g}) is a local model. Pick any positive definite scalar product on \mathfrak{g} . This scalar product determines a Riemannian metric on the total space E of any (H, \mathfrak{g}) -geometry $E \rightarrow M$, the canonical metric of the coframing given by the Cartan connection. Such a Riemannian metric is called a *canonical metric* of the Cartan geometry.

Definition 64. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are Cartan geometries with those local models. A Φ -development is a choice of

- (1) manifold X and
- (2) map $f_0 : X \rightarrow M_0$ and
- (3) map $f_1 : X \rightarrow M_1$ and
- (4) H_0 -equivariant map $F : f_0^* E_0 \rightarrow f_1^* E_1$ and
- (5) F must satisfy $F^* \omega_1 = \Phi \omega_0$.

Definition 65. If a given map $f_0 : X \rightarrow M_0$ has a Φ -development, we say that it Φ -develops.

Definition 66. We will say that *curves Φ -develop freely* from M_0 to M_1 if every curve $f_0 : C \rightarrow M_0$ has a Φ -development.

Lemma 32. The following result holds in the real or complex analytic categories. Suppose that $\Phi : (H_0, \mathfrak{g}_0) \rightarrow (H_1, \mathfrak{g}_1)$ is a local model morphism. Suppose that $E_0 \rightarrow M_0$ and $E_1 \rightarrow M_1$ are Cartan geometries with those local models.

Suppose that we have a curve $f_0 : C \rightarrow M_0$, and C is simply connected. (In the complex case, we add the assumption that $C \neq \mathbb{CP}^1$.) Then the bundle $f_0^* E_0 \rightarrow C$ is trivial. Let s be a section. Let $\iota : f_0^* E_0 \rightarrow E_0$ be the obvious map. The following are equivalent:

- (1) f_0 will Φ -develop in the sense of Cartan geometries,
- (2) the map $\iota \circ s : C \rightarrow E_0$ will Φ -develop in the sense of coframings,
- (3) there is some section s_0 of $f_0^* E_0 \rightarrow C$ so that the map $\iota \circ s_0 : C \rightarrow E_0$ will Φ -develop in the sense of coframings.

Proof. Clearly $f_0^* E_0 \rightarrow C$ must be trivial (holomorphically trivial in the complex analytic category by classification of Riemann surfaces and of the principal bundles on the disk and complex affine line).

Suppose (1): that we can develop f_0 to maps $f_1 : C \rightarrow M_1$ and $F : f_0^*E_0 \rightarrow f_1^*E_1$. Let $\iota_0 : f_0^*E_0 \rightarrow E_0$ and $\iota_1 : f_1^*E_1 \rightarrow E_1$ be the obvious maps. Then $\iota_1 \circ F \circ \iota_0 \circ s : C \rightarrow E_1$ is a Φ -development of s in the sense of coframings. Therefore (1) implies (2) and (3). Clearly (2) implies (3).

Suppose (3): that some section s_0 develops in the sense of coframings. So suppose it develops to $s_1 : C \rightarrow E_1$, with $s_1^*\omega_1 = \Phi s_0^*\omega_0$. Define a map $F : f_0^*E_0 \rightarrow E_1$ by $F(s_0 h_0) = s_1 \Phi(h_0)$. Denote the map $E_1 \rightarrow M_1$ as $\pi_1 : E_1 \rightarrow M_1$. Define a map $f_1 : C \rightarrow M_1$ by $f_1 = \pi_1 \circ F \circ s_0$. It is easy to check that the maps f_0, f_1, F form a Φ -development. Therefore (3) implies (1). \square

We now generalize and thereby prove theorem 1 on page 4.

Theorem 20. For a Cartan geometry $E_1 \rightarrow M_1$, with Cartan connection ω_1 , modelled on a homogeneous space, the following are equivalent:

- (1) some canonical metric on E_1 is complete,
- (2) every canonical metric on E_1 is complete,
- (3) the coframing ω_1 is complete,
- (4) curves I -develop to M_1 from the model,
- (5) curves I -develop to M_1 from some Cartan geometry,
- (6) curves Φ -develop to M_1 from any Cartan geometry, for any model morphism Φ .

Proof. By proposition 1 on page 8, (1), (2) and (3) are equivalent. Each of them implies that all curves $C \rightarrow E_0$ develop in the sense of coframings to curves $C \rightarrow E_1$, by proposition 1 again. They are therefore equivalent to existence of developments of all curves by lemma 32 on the facing page, i.e. to (6). Clearly (6) implies (4) and (5). Obviously (4) implies (5). So we have only to show that (5) implies (1). By lemma 32, (5) implies that one can develop all curves from some coframing. By proposition 1, this implies (1). \square

We now prove the complex analytic analogue of theorem 1 on page 4:

We now generalize and thereby prove theorem 2.

Theorem 21. For a holomorphic Cartan geometry $E_1 \rightarrow M_1$, with Cartan connection ω_1 , modelled on a homogeneous space, the following are equivalent:

- (1) some canonical metric on E_1 is complete,
- (2) every canonical metric on E_1 is complete,
- (3) the coframing ω_1 is complete,
- (4) curves I -develop to M_1 from the model,
- (5) curves I -develop to M_1 from some Cartan geometry,
- (6) curves Φ -develop to M_1 from any Cartan geometry, for any model morphism Φ .

Proof. The proof is identical to the proof of theorem 20 with one small problem: if $f_0 : C \rightarrow M_0$ is a curve, and $C = \mathbb{CP}^1$, then we can't directly employ lemma 32. However, at each stage where we need lemma 32, we can first pop off one point of C , and develop the restriction to $C \setminus \text{pt} = \mathbb{C}$. The development is given by holomorphic first order ordinary differential equations, so is unique when it exists. But then we can change our choice of point to pop off, and by uniqueness we will be able to patch the development of this other affine chart together with the first one to give a global development of f_0 . \square

20. COMPACT STRUCTURE GROUP

We will slightly generalize Clifton's results [9] on Euclidean connections.

Definition 67. Suppose that (H, \mathfrak{g}) is a local model and that H is compact. Pick an H -invariant inner product on \mathfrak{g} (the Lie algebra of G). Write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$$

as an orthogonal decomposition in the inner product.

Suppose that $H \rightarrow E \rightarrow M$ is a Cartan geometry with Cartan connection ω . Let E have the canonical metric derived from the inner product on \mathfrak{g} . Then take the orthogonal complement $V^\perp \subset TE$ to the tangent spaces V of the fibers of $E \rightarrow M$. This orthogonal complement is preserved by H -action, since the metric is H -invariant. The projection $\pi : E \rightarrow M$ identifies $V^\perp = \pi^*TM$, and the map $\pi' : V^\perp \rightarrow TM$ is therefore an isometry for a unique Riemannian metric on M , to be called the *canonical metric* on M .

Lemma 33. Suppose that $H \rightarrow E \rightarrow M$ is a Cartan geometry with local model (H, \mathfrak{g}) and that H is compact. Then $E \rightarrow M$ is a Riemannian submersion, i.e. a submersion for which the map $V^\perp \rightarrow TM$ on the orthogonal complement to the fibers is an isometry of vector bundles with metric. In particular, lengths of curves contract under the mapping $E \rightarrow M$.

Corollary 15. If $H \rightarrow E \rightarrow M$ is a (H, \mathfrak{g}) -geometry and H is compact, then any Cartan geometry modelled on (H, \mathfrak{g}) rolls on M just when M is complete in the canonical Riemannian metric.

Proof. The map $E \rightarrow M$ is a Riemannian submersion, so if closed balls in E are compact, they will map to balls in M of the same radius, which will then also be compact. Conversely, if M is complete in the canonical Riemannian metric, then the ball of any given radius about a point of E will map to a ball of the same radius in M , a compact set, so will live in the preimage of that ball. Because the fibers of $E \rightarrow M$ are compact, this preimage will also be compact. \square

21. MONODROMY

Definition 68. Take any connected manifold X . Let $\pi : \tilde{X} \rightarrow X$ be the universal covering map. Suppose that $\phi_0 : X \rightarrow M_0$ is any smooth map, and M_0 and M_1 bear (H, \mathfrak{g}) -geometries. Let $\tilde{\phi}_0 = \phi_0 \circ \pi : \tilde{X} \rightarrow M_0$. Assume that there is a development $\tilde{\phi}_1 : \tilde{X} \rightarrow M_1$. Then we have an isomorphism $\tilde{\phi}_0^*E_0 = \tilde{\phi}_1^*E_1$. The map $\tilde{\phi}_0^*E_0 \rightarrow \phi_0^*E_0$ is a covering map, with covering group $\pi_1(X)$. Moreover, $\pi_1(X)$ acts on $\tilde{\phi}_0^*E_0 = \tilde{\phi}_1^*E_1$ as bundle automorphisms over the deck transformations of \tilde{X} . We refer to this action as the *monodromy* of the development.

Picking a frame $\tilde{e}_0 \in \tilde{\phi}_0^*E_0$ and corresponding $\tilde{e}_1 \in \tilde{\phi}_1^*E_1$, and corresponding points $e_0 \in \phi_0^*E_0$, $\tilde{x} \in \tilde{X}$ and $x \in X$, we will examine the monodromy orbit of \tilde{e}_1 .

Lemma 34. The monodromy of the development is a free and proper action preserving $\tilde{\phi}_0^*\omega$. Two elements of $\pi_1(X)$ have the same monodromy action on all of $\tilde{\phi}_1^*E_1$ just when they have the same monodromy action on some element of $\tilde{\phi}_1^*E_1$.

Proof. If two elements have the same effect on \tilde{e}_1 , then composing one with the inverse of the other produces an element γ fixing \tilde{e}_1 . The action of $\pi_1(X)$ commutes

with the action of H , so γ fixes every element of the H -orbit through \tilde{e}_1 . Moreover, γ fixes $\tilde{\phi}_1^*\omega$.

Take any path $p(t)$ in $\tilde{\phi}_1^*E_1$ with $p(0) = \tilde{e}_1$. Let $A(t) = \dot{p}(t) \lrcorner \tilde{\phi}_1^*\omega$. The only solution $q(t)$ to $\dot{q}(t) \lrcorner \tilde{\phi}_1^*\omega = A(t)$ satisfying $q(0) = \tilde{e}_1$ is $p(t)$. This differential equation and initial condition are γ invariant, and therefore all points of $p(t)$ are fixed by γ . Therefore γ fixes every point in the path component of the H -orbit of \tilde{e}_1 , i.e. γ acts trivially on $\tilde{\phi}_1^*E_1$. \square

Lemma 35. Suppose that $f : X \rightarrow Y$ is a smooth map of manifolds, equivariant for free and proper actions of a group Γ on X and Y . Then

- (1) the quotient spaces $\bar{X} = \Gamma \backslash X$ and $\bar{Y} = \Gamma \backslash Y$ are smooth manifolds,
- (2) the obvious maps $X \rightarrow \bar{X}$ and $Y \rightarrow \bar{Y}$ are Γ -bundles,
- (3) the quotient map $\bar{f} : \bar{X} \rightarrow \bar{Y}$ on Γ -orbits is a smooth map, and
- (4) any smooth local sections of $X \rightarrow \bar{X}$ and $Y \rightarrow \bar{Y}$ identify f and \bar{f} ; in particular

$$\text{rk } f'(x) = \dim \Gamma + \text{rk } \bar{f}'(\bar{x})$$

when $x \in X$ maps to $\bar{x} \in \bar{X}$.

Proof. It is well known that $X \rightarrow \bar{X}$ is a smooth principal bundle for a unique smooth structure on \bar{X} , and by the same token for $Y \rightarrow \bar{Y}$. Clearly \bar{f} is well defined, and continuous, and lifts to f under any smooth local sections of $X \rightarrow \bar{X}$ and $Y \rightarrow \bar{Y}$, so is smooth. \square

Theorem 22. Take any connected manifold X . Suppose that M_0 and M_1 are manifolds bearing G/H -geometries. Take $\phi_0 : X \rightarrow M_0$ any smooth map. Assume that there is a development of some covering space of X to M_1 . Then a (possibly different) covering space $\hat{X} \rightarrow X$ develops from M_0 to M_1 just when the monodromy orbit of $\pi_1(\hat{X})$ on some point $\tilde{e}_1 \in \tilde{\phi}_1^*E_1$ maps to a single point in E_1 .

Proof. We can replace X by \hat{X} if needed to arrange that $X = \hat{X}$ without loss of generality. Clearly X develops from M_0 to M_1 just when $\tilde{\phi}_1$ is $\pi_1(X)$ -invariant. Moreover, if X develops, then the monodromy orbit of $\pi_1(X)$ on any point \tilde{e}_1 must map to a single point in E_1 . Suppose that the monodromy orbit of $\pi_1(X)$ on \tilde{e}_1 maps to a single point of E_1 . Let $x \in X$ be the point of X which is the image of \tilde{e}_1 under the obvious bundle map $\tilde{\phi}_1^*E_1 \rightarrow X$. The map $\tilde{\phi}_1^*E_1 \rightarrow E_1$ is H -equivariant, so every monodromy orbit of $\pi_1(X)$ above x is mapped to a single point of E_1 . Take a point $\tilde{e}_1 \in \tilde{\phi}_1^*E_1$, and suppose that \tilde{e}_1 maps to $e_1 \in E_1$. Take any smooth path $p(t)$ in $\tilde{\phi}_1^*E_1$ with $p(0) = \tilde{e}_1$. Let $A(t) = \dot{p}(t) \lrcorner \tilde{\phi}_1^*\omega$. The only solution $q(t)$ to $\dot{q}(t) \lrcorner \tilde{\phi}_1^*\omega = A(t)$ satisfying $q(0) = \tilde{e}_1$ is $p(t)$. Therefore in E_1 , the only solution $q(t)$ to $\dot{q}(t) \lrcorner \omega = A(t)$ satisfying $q(0) = e_1$ is the image in E_1 of $p(t)$. The monodromy group will move \tilde{e}_1 around the monodromy orbit, moving $p(t)$ to another curve, but yielding the same solution curve $q(t)$ in E_1 . Therefore the map $\tilde{\phi}_1^*E_1 \rightarrow E_1$ is invariant under $\pi_1(X)$ and drops to a smooth map on the quotient: $\pi_1(X) \backslash \tilde{\phi}_1^*E_1 \rightarrow E_1$.

The development $\tilde{\phi}_0^*E_0 = \tilde{\phi}_1^*E_1$ identifies $\phi_0^*E_0 = \pi_1(X) \backslash \tilde{\phi}_0^*E_0 = \pi_1(X) \backslash \tilde{\phi}_1^*E_1$. By equivariance under the action of H , our map descends to a smooth map $\phi_1 : X = \pi_1(X) \backslash \tilde{\phi}_1^*E_1 / H \rightarrow E_1 / H = M_1$. Define a map $\Phi : \phi_0^*E_0 \rightarrow \phi_1^*E_1$ by quotienting $\tilde{\Phi}$ to $\pi(X)$ orbits. By lemma 35, Φ is a local diffeomorphism, an H -bundle isomorphism, and satisfies $\omega_0 = \omega_1$, so a development. \square

22. ROLLING ALONG LIPSCHITZ CW-COMPLEXES

Definition 69. A locally Lipschitz or C^k or smooth or analytic (etc.) CW-complex is a CW-complex whose attaching maps are locally Lipschitz or C^k or smooth or analytic (etc.). A map between CW-complexes is locally Lipschitz or C^k or smooth or analytic (etc.) just when all of its restrictions to each simplex are.

For example, a manifold or real or complex analytic variety, possibly with boundary and corners, is a locally Lipschitz CW-complex. Clearly the proofs above only require that X be connected by Lipschitz curves, so hold equally well with X any connected locally Lipschitz CW-complex, rather than a manifold, the developed map ϕ_0 locally Lipschitz, and the isomorphism Φ locally essentially bounded. In particular, in theorem 1 on page 4, the curves we roll on can have arbitrary analytic singularities. One could instead try to uniformize the curve first, but then it would not be so clear that the development could be “deuniformized”. Locally Lipschitz development is likely to be useful in developing calibrated cycles in studying Cartan geometries modelled on symmetric spaces.

23. APPLICATIONS

Theorem 23. Suppose that M_0 and M_1 are manifolds and bear real/complex analytic G/H -geometries. Suppose that the model G/H rolls freely on M_1 . Every local development of a real/complex analytic map $\phi_0 : X \rightarrow M_0$ from a simply connected analytic variety X extends to a real/complex analytic development $\phi_1 : X \rightarrow M_1$. Moreover $\phi_0^*TM_0 = \phi_1^*TM_1$ are isomorphic vector bundles on X .

Proof. Clearly it is enough to extend the development along all real curves in X . The local obstructions vanish by analyticity. The vector bundle isomorphism follows from lemma 1 on page 3. \square

Corollary 16. Let M be a complex manifold bearing a complex analytic Cartan geometry modelled on a homogeneous space G/H , and that curves in the model roll on M . Each point of M lies in the image of a holomorphic map $\mathbb{C} \rightarrow M$. Moreover M contains a rational curve through every point just if the model G/H contains a rational curve, and otherwise M contains no rational curves.

For example, Kobayashi hyperbolic complex manifolds admit no complete holomorphic Cartan geometries.

Proof. We need only prove that G has a complex subgroup whose orbit in G/H is a complex curve. This is easy to see if G contains a semisimple group, since G then contains $\mathrm{SL}(2, \mathbb{C})$, and we can just examine the homogeneous spaces of $\mathrm{SL}(2, \mathbb{C})$ by examining closed subgroups. If G contains no semisimple group, then G is solvable and contains a complex abelian subgroup of positive dimension, and the result is obvious. \square

24. IDEAS FOR FURTHER RESEARCH

Lebrun and Mason [27] found a twistor approach to construct Zoll surfaces as moduli spaces of pseudoholomorphic disks. It might be possible to generalize this for parabolic geometries associated to split form semisimple Lie groups.

Suppose that morphism curvature is valued in a representation V , and that $V \subset \mathrm{Sym}^2(W)$ or $V \subset W^* \otimes W$ for some representation W . Then we can consider

the signs of eigenvalues of morphism curvature. This might lead to Toponogov triangle theorems for Cartan geometries.

One can define morphism graphs, a directed graph with homogeneous spaces at vertices and morphisms at edges, with any two paths with the same end points representing the same composition of morphisms. This is a natural way to look at development, in the context of generalized Cartan geometries. A development is really a morphism between morphisms of generalized Cartan geometries.

In the complex analytic setting, perhaps morphisms from a rational homogeneous variety would have unobstructed deformation theory, and perhaps all holomorphic map deformations are morphism deformations.

It would be nice to know which 2-plane fields on 5-manifolds come from rolling one surface on another. More generally, one naturally wants to understand how much surface geometry is encoded in the 2-plane field associated to a pair of surfaces.

It would help to be able to say which morphisms Φ of homogeneous spaces have the property that every morphism modelled on Φ will have vanishing morphism curvature, in particular, for Φ a morphism of rational homogeneous varieties.

Suppose that G is the split real form of a semisimple Lie group, and $P \subset G$ is a parabolic subgroup. Suppose that G has Lie algebra \mathfrak{g} , and suppose that \mathfrak{g} is filtered, with $\mathfrak{p} = \mathfrak{g}^0$ the 0-degree part. (Every parabolic Lie subalgebra arises in this way.) Take any G/P -geometry, say $P \rightarrow E \rightarrow M$. The filtration induces a P -module filtration of $\mathfrak{g}/\mathfrak{p}$,

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_N = \mathfrak{g}/\mathfrak{p}$$

for which V_1 is a direct sum of irreducible representations; see [2, 5]. This imposes a vector bundle $E \times_P V_1 \subset TM$. Let P^c be the maximal compact subgroup of P . Then $E \rightarrow M$ has a principal P^c -subbundle, say $E^c \rightarrow M$. (There are many such subbundles, so pick one.) Then $E^c \rightarrow M$ imposes a canonical Riemannian metric on M as in definition 67 on page 56.

Conjecture 6. If this Riemannian metric on M has positive Ricci curvature along $E \times_P V_1$, then the G/P -geometry is complete.

If proven, this conjecture will apply to nondegenerate 2-plane fields.

There is nothing known about Cartan geometries on Banach manifolds. Some of these theorems might hold in that wider context.

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